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Goals

- $\det A = \det A^T$
- det is multilinear & alternating in columns as well
- Compute det with row + col ops.
- Permutation matrices
- Sign of a permutation

Defn $A \in F^{n \times n}$ is an elementary matrix when it is obtained from I_n by a single row operation.

E.g. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Question Let $E \in F^{n\times n}$ be an elementary matrix corresponding to a particular row operation. How can you describe EA for $A \in F^{n\times m}$?

Guess:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$$

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+\lambda c & b+\lambda d \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ c & d \end{pmatrix}$$

Answer (moral exc) It implements E's corresponding row op on A.

Note By G-J reduction, $\exists$ elementary matrices $E_1, \dots, E_\ell$ s.t. $\text{rref}(A) = E_\ell E_{\ell-1} \cdots E_2 E_1 A$.

Thm For all $A \in F^{n \times n}$, $\det A = \det A^T$.

E.g.

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det \begin{pmatrix} a & c \\ b & d \end{pmatrix} = ad - cb$$

For the general case, we need some additional facts.

$$\boxed{\textbf{Thm} \text{ For all } A, B \in F^{n \times n},\ \det(AB) = \det(A)\det(B).}$$

Pf Upcoming HW! $\square$

det is multiplicative

$\det(A+B) \neq \det A + \det B$

Prop (a) For $A \in F^{l \times m}$, $B \in F^{m \times n}$, $(AB)^T = B^T A^T$.

(b) For $A \in F^{n \times n}$ invertible, $(A^T)^{-1} = (A^{-1})^T$.

Pf (a) is part of your hw.

Let $f = \mathrm{map}_A = A\cdot$

So $f^{-1} = \mathrm{map}_{A^{-1}} = A^{-1}\cdot$

(b) $\mathrm{id}_{F^n} = f \circ f^{-1} \overset{(\ )^*}{\Longrightarrow} \underset{(F^n)^*}{\mathrm{id}} = \overset{(\mathrm{id}_{F^n})^* \ \shortparallel}{\mathrm{id}^*_{F^n}} = (f^{-1})^* \circ f^*$

$\underset{\text{take mat's}}{\Longrightarrow}$ $I_n = (A^{-1})^T A^T$. $\square$

so $(A^{-1})^T = (A^T)^{-1}$.

Lemma Let E be an elementary matrix. Then $\det E = \det E^T \neq 0$.

Pf (1) If $I_n \xrightarrow{r_i \leftrightarrow r_j} E$, then $E \overset{\text{symmetric}}{=} E^T$ and $\det E = -1 = \det E^T$.

(2) If $I_n \xrightarrow{r_i \to \lambda r_i} E$, then $E = E^T$ and $\det E = \lambda = \det E^T$.

(3) If $I_n \xrightarrow{r_i \to r_i + \lambda r_j} E$ for $i \neq j$, then

$$I_n \xrightarrow{r_j \to r_j + \lambda r_i} E^T \quad \text{and} \quad \det E = 1 = \det E^T. \quad \square$$

Pf that $\det A = \det A^T$ First suppose $\mathrm{rref}(A) \neq I_n$. Then $\mathrm{rank}(A)$ $= \mathrm{rank}(A^T) < n$, so $\det A = 0 = \det A^T$. Now assume

row rank = col rank!

$\mathrm{rref}(A) = I_n$. Take elementary matrices $E_1, \ldots, E_\ell$ s.t.

$$I_n = E_\ell \cdots E_1 A$$

$$\overset{\det(\,)}{\Longrightarrow} \quad 1 = \det(E_\ell) \cdots \det(E_1) \det A$$

$(\,)^T$

Also $I_n = I_n^T = (E_\ell \cdots E_1 A)^T = A^T E_1^T \cdots E_\ell^T$

$$\overset{\det(\,)}{\Longrightarrow} \quad 1 = \det(A^T) \det(E_1^T) \cdots \det(E_\ell^T)$$

Hence $\det A = \det(A^T)$.

Note Target of det is F which is commutative! And det multi'v!

Cor det is multilinear & alternating as a function of the columns of the input matrix.

Pf $(\ )^T$ swaps rows & columns and $\det A = \det A^T$. $\square$

E.g. $\det \begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 2 \\ 3 & 1 & 3 \end{pmatrix} = 0$ b/c $1^{st}, 3^{rd}$ columns are equal + det alternating in cols.

Defn A <u>permutation</u> of a set X is a bijection $\sigma : X \to X$. If τ is another permutation, then so is $\sigma \circ \tau$. The set $\mathfrak{S}_X$ of permutations of X together with the binary operation $\circ$

is called the symmetric group of X. If $X = \{1, 2, \dots, n\}$, then we denote this by $\mathfrak{S}_n$.

Math 332: $(G, \cdot)$ is a set G + binary op $\cdot : G \times G \to G$ s.t. $\cdot$ is assoc, $\exists$ identity for $\cdot$, and two-sided inverses for $\cdot$ exist.

E.g. There are six elements of $\mathfrak{S}_3$:

In general, $|\mathfrak{S}_n| = n!$.

Defn For $\sigma \in \mathfrak{S}_n$, the permutation matrix corresponding to σ is $P_\sigma \in F^{n \times n}$ with i-th row $e_{\sigma(i)}$. I.e., P_σ is obtained from I_n by permuting its columns according to σ.

E.g.

$$P_\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \overbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}^{e_{\sigma(2)}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = e_2$$

Check (1) If A has rows $r_1, \dots, r_n$ then $P_\sigma A$ has i-th row $r_{\sigma(i)}$.

(2) $P_\sigma e_{\sigma(i)} = e_i$

(3) $P_\sigma P_\tau = P_{\tau \circ \sigma}$ Order of σ, τ swaps.

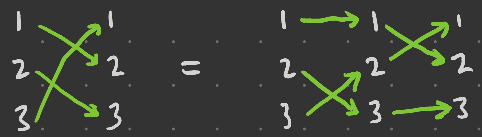
Defn The **sign** of a permutation $\sigma \in \mathfrak{S}_n$ is

$$\operatorname{sgn}(\sigma) = \det(P_\sigma) = \pm 1 .$$

If $\operatorname{sgn}(\sigma) = 1$, call σ **even**; if $\operatorname{sgn}(\sigma) = -1$, call σ **odd**.

Fact (Math 332) Every permutation is a composition of transpositions, so P_σ is obtained from I_n by some number of column swaps. Each swap changes det by a factor of -1 (and $\det I_n = 1$).

Thus $\operatorname{sgn}(\sigma) = (-1)^{\underbrace{\#\text{transpositions for } \sigma}_{\text{takes many values, but all have same parity}}} \in \{\pm 1\}$

E.g.  is even.

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Thm (Permutation expansion) For $A \in F^{n\times n}$,

$$\det A = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{n\sigma(n)}$$

Thus $\det A$ is a homogeneous degree n polynomial in the entries of A.