24.X.14

Goals

- $\det : F^{n\times n} \longrightarrow F$ as an alternating, multilinear function such that $\det(I_n) = 1$.
- Compute det via row ops
- $\det A \neq 0 \iff \operatorname{rank} A = n \iff A$ invertible

Hunting for a function $\det : F^{n\times n} \longrightarrow F$ with some special properties. If $A \in F^{n\times n}$ has rows $r_1, \dots, r_n \in F^n$, write $\det A = \det(r_1, \dots, r_n)$.

$$\det(\lambda r_1 + r_1', r_2, \dots, r_n) = \lambda \det(r_1, \dots, r_n) + \det(r_1', \dots, r_n)$$

Desiderata:

Multilinear: det is linear in each of its variables

Alternating: If $r_i = r_j$ for some $i \neq j$, then $\det(r_1, \dots, r_n) = 0$

Normalized: $\det(I_n) = \det(e_1, \dots, e_n) = 1$.

Thm For each $n \geq 0$, there is a unique determinant function satisfying these properties.

Proof — later! For now, derive consequences of properties.

Prop [det & row ops]

(1) If $A \xrightarrow{r_i \leftrightarrow r_j} B$, then $\det B = -\det A$] Alternative defn of alternating — but doesn't work for char $F = 2$ e.g. $\mathbb{Z}/2\mathbb{Z}$

(2) If $A \xrightarrow{r_i \to \lambda r_i} B$, then $\det B = \lambda \det A$

(3) If $A \xrightarrow{r_i \to r_i + \lambda r_j} B$, then $\det B = \det A$.

Pf Sketch (1)
$$
\begin{aligned}
0 &= \det(r_1 + r_2, r_1 + r_2) \\
&= \cancel{\det(r_1, r_1)} + \det(r_1, r_2) + \det(r_2, r_1) + \cancel{\det(r_2, r_2)} \\
&= 0 + \det A + \det B + 0 = \det A + \det B.
\end{aligned}
$$

(2) Multilinearity.

(3) $\det(r_1, \lambda r_1 + r_2) = \lambda \cancel{\det(r_1, r_1)} + \det(r_1, r_2)$

$= 0 + \det(r_1, r_2)$. $\square$

<u>E.g.</u> $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det((a,b), (c,d))$

$= a\det(e_1, (c,d)) + b\det(e_2, (c,d))$

$= ac\,\cancel{\det(e_1, e_1)} + ad\det(e_1, e_2)$

$+ bc\det(e_2, e_1) + bd\,\cancel{\det(e_2, e_2)}$

$= ad \cdot \det I_2 - bc \cdot \det I_2$

$= ad - bc$.

Note In general, we can use G-J reduction to compute determinants.

E.g.

$$\det(A) = \det\begin{pmatrix} 1 & 2 & -2 \\ 9 & 4 & 0 \\ 2 & 2 & 4 \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & 2 & -2 \\ 0 & -14 & 18 \\ 0 & -2 & 8 \end{pmatrix}$$

$$= -\det\begin{pmatrix} 1 & 2 & -2 \\ 0 & -2 & 8 \\ 0 & -14 & 18 \end{pmatrix}$$

$$= 2\det\begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & -4 \\ 0 & -14 & 18 \end{pmatrix}$$

$$= 2\det\begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & -4 \\ 0 & 0 & -38 \end{pmatrix}$$

$$= 2(-38)\det\begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= 2(-38)\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= 2(-38) = -76.$$

E.g.

$$\det\begin{pmatrix} 4 & 2 & -1 & 8 \\ 0 & 5 & 1 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$= 4\cdot 5\cdot 2\cdot 3 \det\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= 120 \det I_4$$

$$= 120$$

Prop The determinant of an upper triangular matrix

$$\begin{pmatrix} * & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & & * \end{pmatrix} = \begin{pmatrix} a_1 & & & * \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix}$$

is $a_1 \cdots a_n$, the product of its diagonal entries.

Pf The above argument generalizes as long as there are no 0's on the diagonal. In general, $\det A = \lambda \det(\mathrm{rref}(A))$ for some $\lambda \in F \setminus \{0\}$ by G-J reduction. If A has a 0 on its diagonal, then $\mathrm{rref}(A)$ has a row of all 0's and — pulling a 0 scalar out — we get $\det A = 0$. □

Multilinearity $\quad U, V, W \quad F$-vs's

$f: V \times W \longrightarrow U$ is bilinear when

each function $f|_{V \times \{w\}} : V \longrightarrow U$

$f|_{\{v\} \times W} : W \longrightarrow U$

is linear.

Multilinear is this with more variables: $f: V_1 \times \cdots \times V_n \longrightarrow U$.

For det, write $F^{n \times n} = \underbrace{F^n \times F^n \times \cdots \times F^n}_{n \text{ times}}$

Prop For $A \in F^{n\times n}$, TFAE:

(1) $\det A \neq 0$, (2) $\operatorname{rank} A = n$, (3) A is invertible.

Pf Already know (2) $\Leftrightarrow$ (3), so suffices to check (1) $\Leftrightarrow$ (2).

Since $\det A = \lambda \det(\operatorname{rref}(A))$ for some $\lambda \in F - \{0\}$, we know $\det A = 0$ $\Leftrightarrow$ $\det(\operatorname{rref}(A)) = 0$. The rank of A is n iff $\operatorname{rref}(A) = I_n$ iff $\det A = \lambda \neq 0$. $\square$

when $\operatorname{rref}(A) \neq I_n$, it is upper triangular with a row of 0's so $\det = 0$.

Preview

(1) $\det A^T = \det A$

(2) $\det AB = \det A \cdot \det B$

(3) "Laplace expansion" of det along any row or column

(4) "Permutation expansion"

sign $= \pm 1$

$$\det A = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\, A_{1\sigma(1)} \cdots A_{n\sigma(n)}$$

$n!$ terms

`\mathfrak{S}_n`

thus $\det A$ is a homogeneous degree n polynomial in entries of A

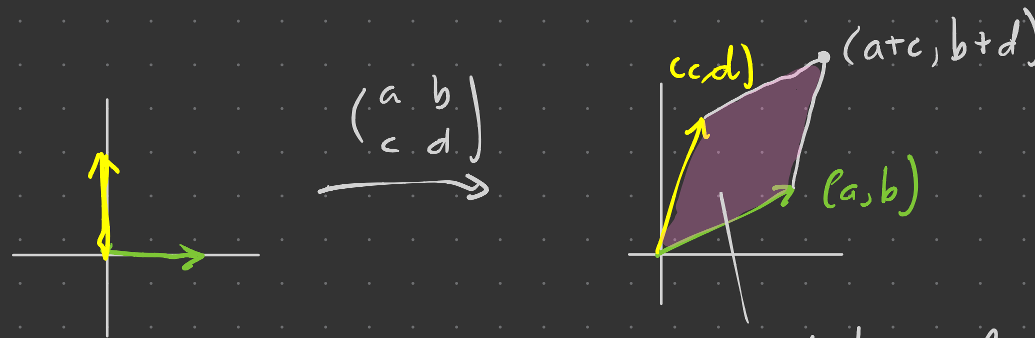
(5) Over $\mathbb{R}$, $\det A$ = "signed volume" of the parallelipiped spanned by the columns of A.

(6) det exists and is unique!

$\mathfrak{S}_n = \{\sigma : \underline{n} \to \underline{n} \mid \sigma \text{ bij}\}$

$\underline{n} = \{1, 2, \ldots, n\}$

$n=2$



parallelipiped spanned by col's of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

<u>Claim</u> $\text{vol}(\quad) = \left|\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right|$

$= |ad-bc|$

Get $-1\cdot$ the volume when A 'reverses orientation':

$$\det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0\cdot 0 - 1\cdot 1 = -1$$