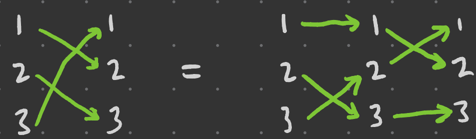
Thus $\operatorname{sgn}(\sigma) = (-1)^{\underbrace{\#\text{transpositions for } \sigma}_{\text{takes many values, but all have same parity}}} \in \{\pm 1\}$

E.g.



is even.

24. $\overline{\text{X}}$. 18

Thm (Permutation expansion) For $A \in F^{n \times n}$,

$$\det A = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{n\sigma(n)}$$

Thus $\det A$ is a homogeneous degree n polynomial in the entries of A.

E.g.

$1 \to 1,\ 2 \to 2,\ 3 \to 3$	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$	$a_{11}a_{22}a_{33}$
$1 \to 2,\ 2 \to 1,\ 3 \to 3$	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$	$-a_{12}a_{21}a_{33}$
$1 \to 3,\ 2 \to 2,\ 3 \to 1$	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$	$-a_{13}a_{22}a_{31}$
$1 \to 1,\ 2 \to 3,\ 3 \to 2$	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$	$-a_{11}a_{23}a_{32}$
$1 \to 2,\ 2 \to 3,\ 3 \to 1$	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$	$a_{12}a_{23}a_{31}$
$1 \to 3,\ 2 \to 1,\ 3 \to 2$	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$	$a_{13}a_{21}a_{32}$

so $\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ = sum of

Exc Check that permutation expansion gives $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$.

<u>Pf of Thm</u> We want to compute

$$\det A = \det(A_{11}e_1 + A_{12}e_2 + \cdots + A_{1n}e_n, \ \ldots, \ A_{n1}e_1 + A_{n2}e_2 + \cdots + A_{nn}e_n).$$

Expanding by multilinearity, we get n^n terms that look like

$$A_{1j_1} A_{2j_2} \cdots A_{nj_n} \det(e_{j_1}, e_{j_2}, \ldots, e_{j_n}).$$

But if $j_k = j_\ell$ for any $k \neq \ell$, then e_{j_k} will be duplicated in the det expression, which will thus be 0 by the alternating property.

As such, the only possible contributors to $\det A$ are of the form

$$A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{n\sigma(n)} \det(\underbrace{e_{\sigma(1)}, e_{\sigma(2)}, \ldots, e_{\sigma(n)}}_{\text{rows of } P_\sigma})$$

$$= \operatorname{sgn}(\sigma) \prod_{i=1}^{n} A_{i\sigma(i)}$$

for $\sigma \in \mathfrak{S}_n$. $\square$

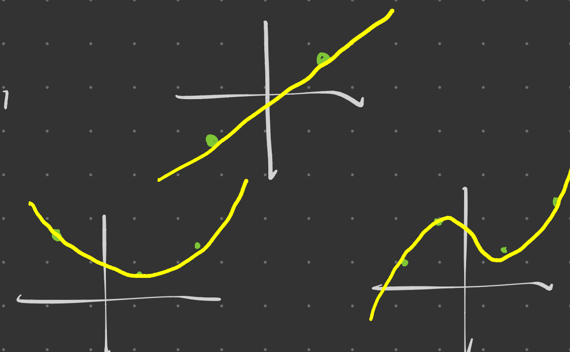
Polynomial interpolation

Suppose we have n points $(x_1, y_1), \dots, (x_n, y_n) \in \mathbb{R}^2$.

We expect there is a degree $n-1$ polynomial interpolating between the points:

$$p(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$$

with $p(x_i) = y_i$ for $1 \le i \le n$.



To find $a_0, \dots, a_{n-1}$, consider the augmented matrix

$$\left(\begin{array}{ccccc|c} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & y_1 \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} & y_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & y_n \end{array}\right).$$

Vandermonde matrix $V \in \mathbb{R}^{n \times n}$

The system $p(x_i) = y_i$, $1 \le i \le n$, has a solution iff $\det V \neq 0$.

Can we compute $\det V$? Yes — via change-of-basis!

Consider the linear transformation

$$f : \mathbb{R}[x]_{\leq n-1} \longrightarrow \mathbb{R}^n$$
$$p(x) \longmapsto (p(x_1), \dots, p(x_n)).$$

Let $\mu = (1, x, \dots, x^{n-1})$, $\mathcal{E} = (e_1, \dots, e_n)$.

Then $A^{\mathcal{E}}_{\mu}(f) = V$.

Now consider a new ordered basis

$$\alpha = (1, x - x_1, (x-x_1)(x-x_2), \dots, (x-x_1)(x-x_2)\cdots(x-x_{n-1}))$$

of $\mathbb{R}[x]_{\leq n-1}$.

Check Since the i-th term of α is monic of degree i (i.e. x^i + (lower order terms)), $A^\mu_\alpha(\mathrm{id}_{\mathbb{R}[x]_{\leq n-1}})$ is upper triangular with 1's on the diagonal.

Thus $\det A^\mu_\alpha(\mathrm{id}_{\mathbb{R}[x]_{\leq n-1}}) = 1$.

We have $A^\varepsilon_\alpha(f) = A^\varepsilon_\mu(f)\, A^\mu_\alpha(\mathrm{id}) = V \cdot A^\mu_\alpha(\mathrm{id})$

$$\Rightarrow \det A^\varepsilon_\alpha(f) = \det V.$$

Evaluating the α polynomials at $x_1, \dots, x_n$, we also have

$$A^{\xi}_{\alpha}(f) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & x_2 - x_1 & 0 & \cdots & 0 \\ 1 & x_3 - x_1 & (x_3 - x_2)(x_3 - x_1) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n - x_1 & (x_n - x_1)(x_n - x_2) & \cdots & (x_n - x_1)(x_n - x_2)\cdots(x_n - x_{n-1}) \end{pmatrix}$$

which is lower triangular with determinant the product of its diagonal entries. Thus

$$\boxed{\det V = \prod_{1 \leq i < j \leq n} (x_j - x_i)} ,$$

We have $\det V = 0$ iff some $x_i = x_j$, $i \neq j$.

If $\det V \neq 0$, then $V^{-1}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a_0 \\ \vdots \\ a_{n-1} \end{pmatrix}$ where

$p(x) = a_0 + a_1 x + a_{n-1} x^{n-1}$ interpolates between the points (x_i, y_i).

Note $(\det V)^2 =:$ discriminant of a polynomial p with distinct roots $x_1, \dots, x_n$ — important in Galois theory.