<u>Goals</u> · Matrix algebra 24. X. 9

· Matrix inverses

Recall the field axioms:

$(F, +, \cdot)$ a field means:

+ is associative, commutative, has an identity 0, and F has additive inverses: $a + (-a) = 0$

· is associative, commutative, has an identity 1, and $F \setminus \{0\}$ has multiplicative inverses: $a \cdot a^{-1} = 1$ for $a \in F \setminus \{0\}$

· distributes (on either side) over +

$(F^{n \times n}, +, \cdot)$ has all the same properties <u>except</u>

- many nonzero matrices don't have multiplicative inverses
- · is not commutative

Terminology $(F^{n\times n}, +, \cdot)$ is a (non-commutative) **ring**.

Note $0_{n\times n} = 0 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ is the additive identity

$I_n = \begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & 0 \\ & & \ddots & \\ & 0 & & 1 \end{pmatrix}$ is the multiplicative identity.

We can use our dictionary to produce non-invertible matrices by pure thought:

is not a bijection so can't have a compositional/multiplicative inverse!

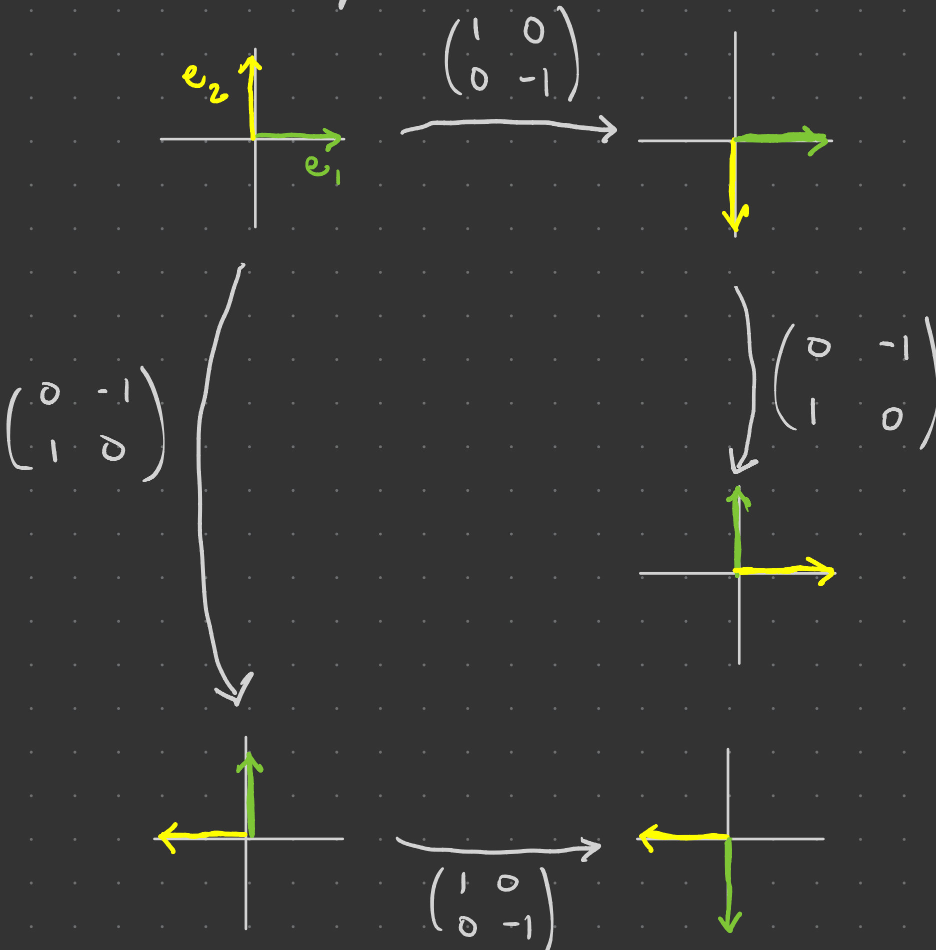
$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{?}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b+d \\ 0 & 0 \end{pmatrix}$$

$$0 \neq 1$$

so never have an inverse.

Similarly for $A \cdot B \neq B \cdot A$:



Check :

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\neq$$

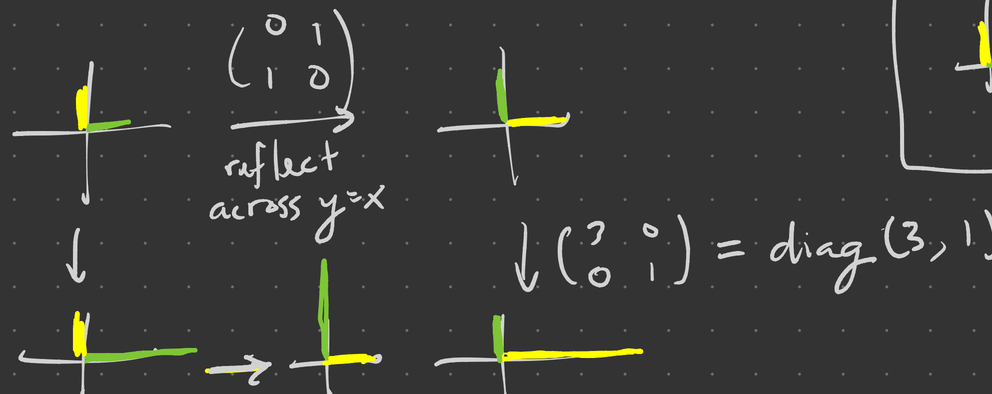
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Problem Find another pair of non-commuting 2×2 matrices.

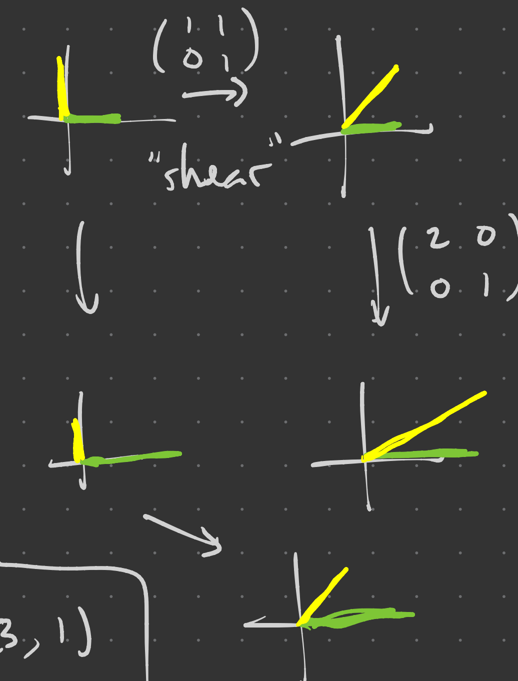
$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}$$

$$\neq$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}$$



$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$



Defn For $A \in F^{n \times m}$, $B \in F^{m \times n}$, when $AB = I_n$

call A the left inverse of B and

B the right inverse of A.

E.g. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Then $AB = I_2$, but $BA \neq I_3$, so A is left (but not right) inverse to B.

Prop For $A, B \in F^{n \times n}$, $AB = I_n$ iff $BA = I_n$.

(Proof deferred.)

Notation $B =: A^{-1}$ when either of these conditions hold.

Thm For $A \in F^{n \times n}$, TFAE: the following are equivalent

(1) A is invertible

(2) $\operatorname{rank}(A) = n$

(3) $\operatorname{rref}(A) = I_n$.

Note (2) $\iff$ (3) via row space / rank results.

Equivalence with (1) will follow from our matrix inversion algorithm:

Let $A = \begin{pmatrix} 0 & 3 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$. To produce a (right) inverse, need to

solve $\begin{pmatrix} 0 & 3 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Equivalently,

$$\begin{pmatrix} 0 & 3 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} a \\ d \\ g \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \iff \begin{aligned} 0a + 3d - g &= 1 \\ 1a + 0d + 1g &= 0 \\ 1a - 1d + 0g &= 0 \end{aligned}$$

$$\begin{pmatrix} 0 & 3 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} b \\ e \\ h \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \iff \begin{aligned} 0b + 3e - 1h &= 0 \\ 1b + 0e + 1h &= 1 \\ 1b - 1e + 0h &= 0 \end{aligned}$$

$$\begin{pmatrix} 0 & 3 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} c \\ f \\ i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \iff \begin{aligned} 0c + 3f - 1i &= 0 \\ 1c + 0f + 1i &= 0 \\ 1c - 1f + 0i &= 1 \end{aligned}$$

So we need to G-J reduce

$$\left(\begin{array}{ccc|c} 0 & 3 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right), \quad \left(\begin{array}{ccc|c} 0 & 3 & -1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \end{array}\right), \quad \left(\begin{array}{ccc|c} 0 & 3 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right).$$

The same row ops put the non-augmented piece in rref in each case, so we can work with the "super-augmented" matrix

$$\left(\begin{array}{ccc|ccc} 0 & 3 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{G\text{-}J} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & 1/4 & 3/4 \\ 0 & 1 & 0 & 1/4 & 1/4 & -1/4 \\ 0 & 0 & 1 & -1/4 & 3/4 & -3/4 \end{array}\right)$$

We learn that $A^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & -1 \\ -1 & 3 & 3 \end{pmatrix}$.

This works in general!

- For $A \in F^{n \times n}$, $(A \mid I_n) \xrightarrow{G\text{-}J} (\mathrm{rref}(A) \mid B)$

- If $\mathrm{rref}(A) = I_n$, then $B = A^{-1}$

- If $\mathrm{rref}(A) \neq I_n$, then $\mathrm{rref}(A)$ has a row of all 0's.
  But B does not as $I_n \xrightarrow{\text{row ops}} B$ so $\mathrm{rank}(B) = n$.
  Thus the system is inconsistent and A has no inverse. $\square$

Now prove $A, B \in F^{n \times n}$, $AB = I_n \Rightarrow BA = I_n$:

Know $\mathrm{map}_A \circ \mathrm{map}_B = \mathrm{id}_{F^n}$ so map_A is surjective $\Rightarrow$ rank $A = n$.

$\Rightarrow$ rank $\mathrm{map}_A = n$. By rank-nullity, $\dim \ker \mathrm{map}_A + n = \dim F^n = n$

$\Rightarrow \dim \ker \mathrm{map}_A = 0 \Rightarrow \ker \mathrm{map}_A = 0 \Rightarrow \mathrm{map}_A$ is also injective!

Thus map_A is bijective and $\exists\, g: F^n \longrightarrow F^m$ function s.t. $g \circ \mathrm{map}_A = \mathrm{id}_{F^n}$. Pre-compose with map_B to get

$$g \circ \mathrm{map}_A \circ \mathrm{map}_B = \mathrm{id}_{F^n} \circ \mathrm{map}_B$$

$$\Rightarrow \quad g \circ \mathrm{id}_{F^n} = \mathrm{map}_B$$

$$\Rightarrow \quad g = \mathrm{map}_B .$$

So $\mathrm{map}_B \circ \mathrm{map}_A = \mathrm{id}_{F^n}$ too $\Rightarrow B \cdot A = I_n$. $\square$

Proof without composition interpretation feels like eldritch magic.