24. IX. 30

Goals
- Define image & kernel
- Rank-nullity theorem

Defn Given a linear transformation $f: V \to W$, the image of f is $\operatorname{im}(f) := fV = \{f(v) \mid v \in V\}$.

Prop $\operatorname{im}(f) \leq W$

Pf (0) We showed $f(0) = 0$ for f linear, so $0 \in \operatorname{im} f$.

(1) Suppose $w_1, w_2 \in \operatorname{im} f$. By defn $\exists v_1, v_2 \in V$ s.t. $f(v_1) = w_1$, $f(v_2) = w_2$. Then $w_1 + w_2 = f(v_1) + f(v_2)$
$= f(v_1 + v_2)$ by linearity so $w_1 + w_2 \in \operatorname{im} f$.

(2) If $w = f(v) \in \operatorname{im} f$, then $\forall \lambda \in F$, $\lambda w = \lambda f(v) = f(\lambda v) \in \operatorname{im} f$. $\square$

Defn The rank of a linear trans'n $f: V \longrightarrow W$ is

$$\operatorname{rank}(f) := \dim \operatorname{im}(f).$$

E.g. Define $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ and linear extension.

$$e_1 \longmapsto (2,1,0)$$
$$e_2 \longmapsto (0,-1,1)$$

$$f(x,y) = f(xe_1 + ye_2)$$
$$= x(2,1,0) + y(0,-1,1) = (2x, x-y, y)$$

Then $\operatorname{im}(f) = \operatorname{span}\{(2,1,0), (0,-1,1)\}$ and $\operatorname{rank}(f) = 2$ (why?)

Rmk If $f: V \longrightarrow W$ linear, $\{b_1, \ldots, b_n\}$ basis of V, then

$$\operatorname{im}(f) = \operatorname{span}\{f(b_1), \ldots, f(b_n)\}.$$

If $f(b_1), \ldots, f(b_n)$ lin dependent, then $\operatorname{rank}(f) < n$.

Recall $f: X \xrightarrow{\text{function}} Y$, $S \subseteq Y$ then the **preimage** of S is

$$f^{-1}S := \{x \in X \mid f(x) \in S\}.$$

$f^{-1}: 2^Y \longrightarrow 2^X$

Prop For $f: V \longrightarrow W$ linear and $U \leq W$, $f^{-1}U \leq V$.

Pf (0) Note $0 \in U$ b/c U subspace and $f(0) = 0$ so $0 \in f^{-1}U$.

(1) Suppose $v_1, v_2 \in f^{-1}U$ so $f(v_1) = u_1$, $f(v_2) = u_2 \in U$.

Thus $f(v_1 + v_2) = f(v_1) + f(v_2) = u_1 + u_2 \in U$ b/c U is a subspace.

Hence $v_1 + v_2 \in f^{-1}U$.

(2) **Exc**. $\square$

Defn For $f: V \longrightarrow W$ linear, the kernel (or nullspace) of f is

$$\ker(f) := f^{-1}\{0\} = \{v \in V \mid f(v) = 0\}$$

Note Since $\{0\} \leq W$, $\ker f \leq V$.

Defn The nullity of f is $\dim \ker f$.

Problem Determine the kernel and nullity of the linear trans'n

$$\begin{aligned} f: \mathbb{R}^2 &\longrightarrow \mathbb{R}^3 \\ e_1 &\longmapsto (2,1,0) \\ e_2 &\longmapsto (0,-1,1) \end{aligned}$$

$(x,y) \in \ker f$

$\Leftrightarrow f(x,y) = 0 \Leftrightarrow x(2,1,0) + y(0,-1,1) = (0,0,0)$

$\Leftrightarrow x = y = 0$ So $\ker(f) = \{0\} \leq \mathbb{R}^2$

E.g. Consider the linear trans'n

$$\frac{d}{dx} : \mathbb{R}[x]_{\leq 2} \longrightarrow \mathbb{R}[x]_{\leq 1}$$

$$a + bx + cx^2 \longmapsto b + 2cx.$$

Then $\ker\left(\frac{d}{dx}\right) = \{ a + bx + cx^2 \mid b + 2cx = 0 \}$

with domain $\mathbb{R}[x]_{\leq 2}$

$$= \{ a \in \mathbb{R}[x]_{\leq 2} \mid a \in \mathbb{R} \}.$$

$$= \operatorname{span}\{1\}$$

and the nullity of $\frac{d}{dx}$ is 1.

The image of $\frac{d}{dx}$ is

apply $\frac{d}{dx}$ to basis of $\mathbb{R}[x]_{\leq 2}$

$$\operatorname{im} \frac{d}{dx} = \operatorname{span}\left\{ \frac{d}{dx}1, \frac{d}{dx}x, \frac{d}{dx}x^2 \right\}$$

$= \mathrm{span}\{0, 1, 2x\}$

$= \mathbb{R}[x]_{\leq 1}$

So $\mathrm{rank}\, \frac{d}{dx} = 2$.

Note $3 = \dim \mathbb{R}[x]_{\leq 2} = \mathrm{rank}\left(\frac{d}{dx}\right) + \mathrm{null}\left(\frac{d}{dx}\right)$

— this is generic.

Thm (rank-nullity) Suppose $f: V \longrightarrow W$ linear, V is finite dim'l. Then

$$\dim V = \mathrm{rank}(f) + \mathrm{null}(f).$$

... { Domain "splits" into kernel and the image.

Pf Suppose $\text{null}(f) = k$ and $\ker(f)$ has basis $\{v_1, \dots, v_k\}$.

We can complete this to a basis $B = \{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ of V.

It suffices to show $\{f(v_{k+1}), \dots, f(v_n)\}$ is a basis of $\text{im}(f)$.

Now $\text{im}(f) = \text{span}\, fB = \text{span}\{f(v_1), \dots, f(v_n)\}$

$= \text{span}\{0, \dots, 0, f(v_{k+1}), \dots, f(v_n)\}$

so $\{f(v_{k+1}), \dots, f(v_n)\}$ generates $\text{im}(f)$.

For linear ind, suppose

$$\lambda_{k+1} f(v_{k+1}) + \dots + \lambda_n f(v_n) = 0.$$

Then $f(\lambda_{k+1}v_{k+1} + \cdots + \lambda_n v_n) = 0$

$\Rightarrow \lambda_{k+1}v_{k+1} + \cdots + \lambda_n v_n \in \ker f$ has basis $\{v_1, \ldots, v_k\}$

$\Rightarrow \exists\, \lambda_1, \ldots, \lambda_k \in F$ s.t.

$$\lambda_1 v_1 + \cdots + \lambda_k v_k = \lambda_{k+1}v_{k+1} + \cdots + \lambda_n v_n .$$

But then $\lambda_1 v_1 + \cdots + \lambda_k v_k - \lambda_{k+1}v_{k+1} - \cdots - \lambda_n v_n = 0$

and by lin ind of $B = \{v_1, \ldots, v_n\}$, get $\lambda_i = 0$ for all i.

In particular, $f(v_{k+1}), \ldots, f(v_n)$ are lin ind. $\square$

Suppose $S = \{v_1, \dots, v_k\}$ lin ind in vs V.

Basis ext'n algorithm:

1. If $\operatorname{span} S = V$, done.
2. O/w take $v_{k+1} \in V \setminus \operatorname{span} S$.
3. Add v_{k+1} to S — $S \cup \{v_{k+1}\}$ is lin ind.
4. Re-index and go back to 1.

If V is fin dim'l, this will terminate after

$\dim V - k$ steps.

E.g. $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ extend linearly

$$e_1 \longmapsto (1,1,1)$$
$$e_2 \longmapsto (-2,0,3)$$
$$e_3 \longmapsto (1,3,6)$$

Find $\ker f := \{(x,y,z) \mid f(x,y,z) = 0\}$

$$= \{(x,y,z) \mid x(1,1,1) + y(-2,0,3) + z(1,3,6) = (0,0,0)\}$$

So need to solve
$$x - 2y + z = 0$$
$$x + 3z = 0$$
$$x + 3y + 6z = 0$$

$$A = \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 3 & 6 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 5 & 5 & 0 \end{array}\right)$$

$\longrightarrow \quad \cdots$

rank $= 2$

$\Rightarrow$ nullity $= 1$.