24.IX.25

Goals
- Define row & column spaces of a matrix
- Use Gauss-Jordan reduction to compute bases for both
- Prove that dimensions of row & column spaces are equal
  $\rightsquigarrow$ def'n of rank of a matrix

Defn
- For $A \in F^{m \times n}$, the row space of A is the span of the rows of A in F^n; the column space of A is the span of the columns of A in F^m.
- The row rank of A is the dimension of its row space; the column rank ______ " ______ column space.

Recall The elementary row operations:

- are linear combinations / reorderings of rows
- are reversible.

Thus $\boxed{\text{row space of } A = \text{row space of } \operatorname{rref}(A)}$

Thm The nonzero rows of $\operatorname{rref}(A)$ form a basis of the row space of A.

Pf Shape! $\square$

E.g. $A = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 3 & 3 & 1 & 0 \\ 7 & 8 & 2 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \boxed{\begin{matrix} 1 & 0 & 2/3 & -4 \\ 0 & 1 & -1/3 & 4 \end{matrix}} \\ \begin{matrix} 0 & 0 & 0 & 0 \end{matrix} \end{pmatrix}$

these are a basis for row space of A

The row rank of A is 2.

Prop Let $A = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \end{pmatrix} \in F^{m \times n}$ with $c_i \in F^m$ (as col vectors).

Let $\tilde{A}$ be any matrix formed by applying row ops to A, and let $\tilde{c}_1, \ldots, \tilde{c}_n$ be the columns of $\tilde{A}$. Then for $\lambda_1, \ldots, \lambda_n \in F$,

$$\underbrace{\sum_{i=1}^{n} \lambda_i c_i = 0}_{①} \iff \underbrace{\sum_{i=1}^{n} \lambda_i \tilde{c}_i = 0}_{②}.$$

Pf We have ① $\iff$

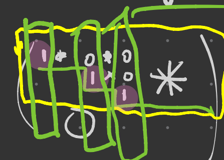
$$\lambda_1 \underbrace{\begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}}_{c_1} + \cdots + \lambda_n \underbrace{\begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}}_{c_n} = 0$$

and sol'ns to this are systems

$$\begin{aligned} a_{11}\lambda_1 + a_{12}\lambda_2 + \cdots + a_{1n}\lambda_n &= 0 \\ &\vdots \\ a_{m1}\lambda_1 + a_{m2}\lambda_2 + \cdots + a_{mn}\lambda_n &= 0. \end{aligned}$$

Row operations don't change sol'n sets, so the result follows. $\square$

<u>Cor</u> Let $E = \mathrm{rref}(A)$ with pivot column indices $j_1, \dots, j_r$. Then the columns of A indexed by $j_1, \dots, j_r$ form a basis of the column space of A.



these cols of A — not of $\mathrm{rref}(A)$

<u>Pf</u> For simplicity, assume $j_1 = 1, \dots, j_r = r$. Then E looks like

$$\begin{array}{c} \begin{pmatrix} 1 & 0 & 0 & * & * & * \\ 0 & 1 & 0 & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{array}{cccccc} E_1 & E_2 & & \cdots & & E_n \end{array} \end{array}$$

in the $m=5$, $n=6$, $r=3$ case.

Let $E_1, \dots, E_n$ denote cols of E,

$A_1, \dots, A_n$ cols of A.

<u>WTS</u> $A_1, \dots, A_r$ are lin ind and generate col space.

Lin ind : Suppose $\lambda_1 A_1 + \cdots + \lambda_r A_r = 0 \overset{\text{prop}}{\Longrightarrow} \lambda_1 E_1 + \cdots + \lambda_r E_r = 0$.

But $E_1, \ldots, E_r$ are lin ind, so $\lambda_1 = \cdots = \lambda_r = 0$.

Generates To show $\operatorname{span}\{A_1, \ldots, A_r\} =$ col space of A, it suffice to show $A_j \in \operatorname{span}\{A_1, \ldots, A_r\}$ for $j > r$.

Since $E_1, \ldots, E_r$ span the col space of E (why?), know

$$\exists \lambda_1, \ldots, \lambda_r \in F \text{ s.t. } \lambda_1 E_1 + \cdots + \lambda_r E_r = E_j$$

$$\iff \lambda_1 E_1 + \cdots + \lambda_r E_r - E_j = 0$$

$$\iff \lambda_1 A_1 + \cdots + \lambda_r A_r - A_j = 0$$

$$\overset{\text{prop}}{\iff} \lambda_1 A_1 + \cdots + \lambda_r A_r = A_j. \qquad \square$$

E.g. $A = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 3 & 3 & 1 & 0 \\ 7 & 8 & 2 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 2/3 & -4 \\ 0 & 1 & -1/3 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

so $\left\{ \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 8 \end{pmatrix} \right\}$ is a basis for the column space of A,

and its column rank is 2. 🤔 Same as row rank?

Thm Row rank of A = column rank of A.

Pf Let $E = \mathrm{rref}(A)$. The # nonzero rows of E = # pivot columns of E.

$\square$

Dfn The <u>rank</u> of A, denoted $\mathrm{rank}(A)$, is its row (or column) rank.

Rank is a unique solution detector:

To compute sol'ns of a homogeneous linear system

$$a_{11}x_1 + \cdots + a_{1n}x_n = 0$$
$$\vdots$$
$$a_{m1}x_1 + \cdots + a_{mn}x_n = 0$$

set $A = (a_{ij}) \in F^{m\times n}$ and compute $\mathrm{rref}(A)$.

The #free variables = #non-pivot columns

$= n - \mathrm{rank}(A)$.

$$\boxed{\exists!\ \text{sol'n} \iff \mathrm{rank}(A) = n}.$$

Note Set of sol'ns is always a subspace of F^n with dimension = #free vars.

For a non-homogeneous system

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$
$$\vdots$$
$$a_{m1}x_1 + \cdots + a_{mn}x_n = b_m$$

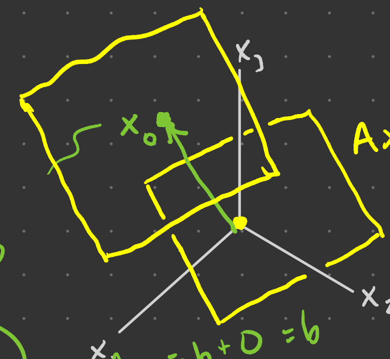
$Ax_0 = b$

if $Ax = 0$

then

$A(x_0 + x)$

$= Ax_0 + Ax = b + 0 = b$



set $A = (a_{ij})$, $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ and compute $\mathrm{rref}((A|b))$.

If the system is consistent, then every sol'n is of the form

$$(\text{particular sol'n}) + (\text{sol'n of } Ax = 0).$$

So if the system is consistent, there is again

$$\boxed{\text{unique sol'n} \iff \mathrm{rank}(A) = n}.$$

E.g. How many sol'ns does

$$\begin{aligned} 2x \qquad\qquad &= 2 \\ 3y + 2z &= 12 \\ y - z &= -1 \end{aligned}$$

have?

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & -1 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 12 \\ -1 \end{pmatrix}$$

$$\frac{5}{3}\lambda = 2$$

$$\lambda = \frac{6}{5}$$

G-J red'n for A:

$$A|b \xrightarrow{r_3 \to r_3 - \frac{1}{3} r_2} \left(\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & 3 & 2 & 12 \\ 0 & 0 & -5/3 & -5 \end{array}\right) \xrightarrow{r_2 \to r_2 + \frac{6}{5} r_3} \left(\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & -5/3 & -5 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$\Rightarrow$ a unique soln to $Ax = b$ iff it's consistent:

$\text{rank}(A) = 3$

I, consistent and sol'n set is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$.

<u>Note</u> col space of A has basis $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$.