$B_n = \{w_1, \dots, w_n\}$ is a basis of V

$\cap$

C In fact, $B_n = C$ b/c o/w $w_{n+1} \in \operatorname{span} B_n$

because otherwise

$\Rightarrow C$ lin dep. $\cancel{\circ}$ $\square$

<u>Cor</u> dimension of fin dim vs's is well-defined

<u>Cor</u> If V is a fin dim vs, $S \subseteq V$ lin ind, then we may extend S to a basis of V by adding some $\dim V - |S|$ elements.

<u>Pf</u> Apply the "basis production algorithm" from the theorem's proof. $\square$

<u>Cor</u> If V fin dim vs and $T \subseteq V$ generates V, then some subset $S \subseteq T$ is a basis of V. $\square$

Cor If $S \subseteq V$ and $|S| = n = \dim V < \infty$, then S is lin ind iff $\operatorname{span} S = V$. $\square$

$e_i = (0, \dots, 0, \overset{i\text{-th}}{1}, 0, \dots, 0)$

E.g. (1) F^n has basis $\{e_1, \dots, e_n\}$ so $\dim F^n = n$.

(2) Let $S = \{(1,0,0), (1,2,0), (1,2,3)\} \subseteq \mathbb{R}^3$. Since

(a) $(1,2,0) \notin \operatorname{span}\{(1,0,0)\}$

(b) $(1,2,3) \notin \operatorname{span}\{(1,0,0), (1,2,0)\}$

(c) $\dim \mathbb{R}^3 = 3 = |S|$

know S is a basis of $\mathbb{R}^3$.

(3) $\dim \underbrace{F[x]_{\leq n}}_{\text{polynomials of deg } \leq n} = n+1$ basis $\{1, x, x^2, \dots, x^n\}$

(4) $\dim F^{m \times n} = mn$.

Problem Determine $\dim F[x,y]_{\leq n}$ where

- $F[x,y]$ = 2-var polynomials over F,
- $\deg \sum \lambda_{ij} x^i y^j = \max\{ i+j \mid \lambda_{ij} \neq 0 \}$
- $F[x,y]_{\leq n} = \{ f \in F[x,y] \mid \deg f \leq n \}$.

24. IX. 23

Condorcet's Voting Paradox

Candidates A, B, C

29 voters

model assumption: every voter can linearly order preferences

Preferences	# voters
$A > B > C$	5
$A > C > B$	4
$B > A > C$	2
$B > C > A$	8
$C > A > B$	8
$C > B > A$	2

$\Rightarrow$

$A > B$: $17 - 12 = 5$

$B > C$: $15 - 14 = 1$

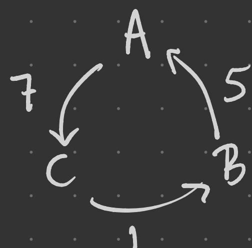
$C > A$: $18 - 11 = 7$

$A > B > C > A$ — Condorcet cycle

Task Teams A, B, C schedule sequential head-to-head votes so your candidate wins — become a dictator through bureaucracy!

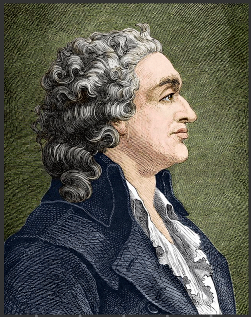
C: first do A vs B — A wins
then A vs C — C wins

We can visualize the total binary preferences as



This is called a Condorcet cycle and it leads to a voting paradox.

Marie Jean Antoine Nicolas de Caritat, Marquis of Condorcet
1743-1794 CE



Goal Use linear algebra to understand how / when such cycles exist.

Set $V = \left\{ \begin{tikzcd} & A \arrow[dl, "c"'] & \\ C \arrow[rr, "b"'] & & B \arrow[ul, "a"'] \end{tikzcd} \;\middle|\; a, b, c \in \mathbb{R} \right\} \cong \mathbb{R}^3$

$\longmapsto (a, b, c)$

An $A > B > C$ voter corresponds to $\begin{tikzcd} & A \arrow[dl, "-1"'] & \\ C \arrow[rr, "1"'] & & B \arrow[ul, "1"'] \end{tikzcd}$, etc

In the above example, the election <u>is</u> the vector

$$5 \cdot \begin{tikzcd} & A \arrow[dl, "-1"'] & \\ C \arrow[rr, "1"'] & & B \arrow[ul, "1"'] \end{tikzcd} + 4 \cdot \begin{tikzcd} & A \arrow[dl, "-1"'] & \\ C \arrow[rr, "-1"'] & & B \arrow[ul, "1"'] \end{tikzcd} + \cdots + 2 \cdot \begin{tikzcd} & A \arrow[dl, "1"'] & \\ C \arrow[rr, "-1"'] & & B \arrow[ul, "-1"'] \end{tikzcd}.$$

$(A > B > C)$ $\quad$ $(A > C > B)$ $\quad$ $(C > B > A)$

<u>Defn</u> Call a vector in $\mathbb{R}^3$ <u>purely cyclic</u> when it is of the form $(\lambda, \lambda, \lambda) = (1,1,1)$ for some $\lambda \in \mathbb{R}$. Let

$C = \operatorname{span}\left\{ \begin{tikzcd} & A \arrow[dl, "1"'] & \\ C \arrow[rr, "1"'] & & B \arrow[ul, "1"'] \end{tikzcd} \right\}$ be the subspace of purely cyclic vectors.

To have <u>no</u> cyclic component is to be perpendicular to C:

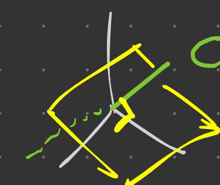
$$(a,b,c) \perp (x,y,z) \iff \underbrace{ax+by+cz}_{(a,b,c)\cdot(x,y,z)} = 0$$

— more on this when we study inner product spaces

So $C^{\perp} = \{(a,b,c) \in \mathbb{R}^3 \mid ak+bk+ck=0 \ \forall k \in \mathbb{R}\}$

$$= \{(a,b,c) \in \mathbb{R}^3 \mid a+b+c=0\}$$

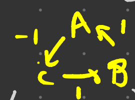
$$= \{b(-1,1,0) + c(-1,0,1) \mid b,c \in \mathbb{R}\}.$$



Get an ordered basis

$$\mathcal{B} = \Big(\underbrace{(1,1,1)}_{C},\ \underbrace{(-1,1,0),\ (-1,0,1)}_{C^{\perp}}\Big) \text{ of } \mathbb{R}^3$$

If $\mathrm{Rep}_B(a,b,c) = (x,y,z)$, call x the cyclic component, y,z the non-cyclic components.



$A > B > C$

E.g. $\mathrm{Rep}_B(1,1,-1) = (x,y,z) \iff (1,1,-1) = x(1,1,1) + y(-1,1,0) + z(-1,0,1)$

$$\iff \begin{array}{l} x - y - z = 1 \\ x + y = 1 \\ x + z = -1 \end{array} \iff \left(\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 2/3 \\ 0 & 0 & 1 & -4/3 \end{array}\right)$$

So $\mathrm{Rep}_B(1,1,-1) = \left(\frac{1}{3}, \frac{2}{3}, \frac{-4}{3}\right)$. 😳 Rational preference $A>B>C$ has a cyclic component!

Similarly, $\mathrm{Rep}_B\ C>B>A = \left(\frac{-1}{3}, \frac{-2}{3}, \frac{4}{3}\right)$.

Defn The sign of the cyclic component is the spin of the vector.

± 1

Positive Spin

cyclic | sum of non-cyclic

$$\begin{tikzcd}[column sep=small] & A \arrow[dl, "-1"'] & \\ C \arrow[rr, "1"'] & & B \arrow[ul, "1"'] \end{tikzcd} = \begin{tikzcd}[column sep=small] & A \arrow[dl, "1/3"'] & \\ C \arrow[rr, "1/3"'] & & B \arrow[ul, "1/3"'] \end{tikzcd} + \begin{tikzcd}[column sep=small] & A \arrow[dl, "-4/3"'] & \\ C \arrow[rr, "2/3"'] & & B \arrow[ul, "2/3"'] \end{tikzcd}$$

A > B > C

$$\begin{tikzcd}[column sep=small] & A \arrow[dl, "1"'] & \\ C \arrow[rr, "1"'] & & B \arrow[ul, "-1"'] \end{tikzcd} = \begin{tikzcd}[column sep=small] & A \arrow[dl, "1/3"'] & \\ C \arrow[rr, "1/3"'] & & B \arrow[ul, "1/3"'] \end{tikzcd} + \begin{tikzcd}[column sep=small] & A \arrow[dl, "2/3"'] & \\ C \arrow[rr, "2/3"'] & & B \arrow[ul, "-4/3"'] \end{tikzcd}$$

B > C > A

$$\begin{tikzcd}[column sep=small] & A \arrow[dl, "1"'] & \\ C \arrow[rr, "-1"'] & & B \arrow[ul, "1"'] \end{tikzcd} = \begin{tikzcd}[column sep=small] & A \arrow[dl, "1/3"'] & \\ C \arrow[rr, "1/3"'] & & B \arrow[ul, "1/3"'] \end{tikzcd} + \begin{tikzcd}[column sep=small] & A \arrow[dl, "2/3"'] & \\ C \arrow[rr, "-4/3"'] & & B \arrow[ul, "2/3"'] \end{tikzcd}$$

C > A > B

Negative Spin

$$\begin{tikzcd}[column sep=small] & A \arrow[dl, "1"'] & \\ C \arrow[rr, "-1"'] & & B \arrow[ul, "-1"'] \end{tikzcd} = \begin{tikzcd}[column sep=small] & A \arrow[dl, "-1/3"'] & \\ C \arrow[rr, "-1/3"'] & & B \arrow[ul, "-1/3"'] \end{tikzcd} + \begin{tikzcd}[column sep=small] & A \arrow[dl, "4/3"'] & \\ C \arrow[rr, "-2/3"'] & & B \arrow[ul, "-2/3"'] \end{tikzcd}$$

C > B > A

$$\begin{tikzcd}[column sep=small] & A \arrow[dl, "-1"'] & \\ C \arrow[rr, "-1"'] & & B \arrow[ul, "1"'] \end{tikzcd} = \begin{tikzcd}[column sep=small] & A \arrow[dl, "-1/3"'] & \\ C \arrow[rr, "-1/3"'] & & B \arrow[ul, "-1/3"'] \end{tikzcd} + \begin{tikzcd}[column sep=small] & A \arrow[dl, "-2/3"'] & \\ C \arrow[rr, "-2/3"'] & & B \arrow[ul, "4/3"'] \end{tikzcd}$$

A > C > B

$$\begin{tikzcd}[column sep=small] & A \arrow[dl, "-1"'] & \\ C \arrow[rr, "1"'] & & B \arrow[ul, "-1"'] \end{tikzcd} = \begin{tikzcd}[column sep=small] & A \arrow[dl, "-1/3"'] & \\ C \arrow[rr, "-1/3"'] & & B \arrow[ul, "-1/3"'] \end{tikzcd} + \begin{tikzcd}[column sep=small] & A \arrow[dl, "-2/3"'] & \\ C \arrow[rr, "4/3"'] & & B \arrow[ul, "-2/3"'] \end{tikzcd}$$

B > A > C

Contribution

$$\begin{tikzcd}[column sep=small] & A \arrow[dl, "-a"'] & \\ C \arrow[rr, "a"'] & & B \arrow[ul, "a"'] \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & A \arrow[dl, "b"'] & \\ C \arrow[rr, "b"'] & & B \arrow[ul, "-b"'] \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & A \arrow[dl, "c"'] & \\ C \arrow[rr, "-c"'] & & B \arrow[ul, "c"'] \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & A \arrow[dl, "-a+b+c"'] & \\ C \arrow[rr, "a+b-c"'] & & B \arrow[ul, "a-b+c"'] \end{tikzcd}$$

The election results in a Condorcet cycle when all three sides of $\begin{tikzcd}[column sep=small] & A \arrow[dl, "-a+b+c"'] & \\ C \arrow[rr, "a+b-c"'] & & B \arrow[ul, "a-b+c"'] \end{tikzcd}$ have the same sign!

If all positive,

$$\begin{aligned} -a+b+c &> 0 \\ a-b+c &> 0 \\ a+b-c &> 0 \end{aligned} \qquad \begin{aligned} 2c > 0 &\Rightarrow c > 0 \\ 2b > 0 &\Rightarrow b > 0 \\ 2a > 0 &\Rightarrow a > 0 \end{aligned}$$

Similarly, if all negative, $a, b, c < 0$.

Thm If there is a Condorcet cycle, then $a, b, c > 0$ or $a, b, c < 0$.

Is the converse true?

$a = 15$, $b = 1$, $c = 1$

$-a+b+c = -13$

$a-b+c = 15$

$a+b-c = 15$

so not a Condorcet cycle

Fact As the # of voters $\to \infty$,

$$P(\text{Condorcet cycle}) \to \frac{\arcsin \frac{\sqrt{6}}{9}}{\pi} \approx 0.0877$$