24. IX. 27

Goals
- define linear transformations
- proving linearity
- determined by action on a basis
- Hom(V, W) as a vector space

or poly diff'l

or diff'l fns

Sets : Functions :: open subsets of $\mathbb{R}$: continuous functions

:: vector spaces : linear transformations

Defn Fix vector spaces V, W over F. A linear transformation (or homomorphism) $L : V \longrightarrow W$ is a function s.t. $\forall u, v \in V$, $\lambda \in F$,

$$L(u+v) = L(u) + L(v) \quad \& \quad L(\lambda u) = \lambda L(u).$$

Equivalently, $L(u+\lambda v) = L(u) + \lambda L(v)$.

Hefferon's "linear transformations" have the same domain & codomain: $L: V \to V$. These are typically called linear endomorphisms.

E.g. $L: V \to W$, $v \mapsto 0$ is the trivial lin trans

E.g. $f: \mathbb{R}^3 \to \mathbb{R}^2$, $(x,y,z) \mapsto (2x+3y,\ x+y-3z)$ is linear.

Exc What assignment on std basis of $\mathbb{R}^3$ determines this lin trans?

$e_1 \mapsto (2,1)$
$e_2 \mapsto (3,1)$
$e_3 \mapsto (0,-3)$

Pf Let $(x,y,z), (x',y',z') \in \mathbb{R}^3$, $\lambda \in \mathbb{R}$. Then

$$f((x,y,z) + \lambda(x',y',z')) = f(x+\lambda x',\ y+\lambda y',\ z+\lambda z')$$
$$= (2(x+\lambda x') + 3(y+\lambda y'),\ (x+\lambda x') + (y+\lambda y') - 3(z+\lambda z'))$$

$= (2x+3y, x+y-3z) + (2\lambda x'+3\lambda y', \lambda x'+\lambda y'-3\lambda z')$

$= (2x+3y, x+y-3z) + \lambda(2x'+3y', x'+y'-z')$

$= f(x,y,z) + \lambda f(x',y',z')$

so f is linear. $\square$

E.g. $\frac{d}{dx} : \underbrace{C^1(\mathbb{R})}_{\text{ctsly diff'l fns } \mathbb{R}\to\mathbb{R}} \longrightarrow \underbrace{C(\mathbb{R})}_{\text{cts fns } \mathbb{R}\to\mathbb{R}}$ is linear. i.e. $\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$

$\frac{d}{dx}(\lambda \cdot f) = \lambda \frac{df}{dx}$

E.g. $\mathbb{R} \longrightarrow \mathbb{R}$ is not linear. Why?
$x \longmapsto x^2$

$-1 = -1\cdot 1 \longmapsto (-1)^2 = 1 \neq -1\cdot(1)^2$ | $1+2 \longmapsto 9 \neq 1^2 + 2^2$

Prop If $L : V \to W$ is linear, then $L(0) = 0$.

Pf By linearity, $L(0_V) = L(0_F \cdot 0_V) = 0_F\, L(0_V) = 0_W$. $\square$

Let $F = \mathbb{Z}/p\mathbb{Z}$. Write $\overset{\text{Frobenius}}{\text{Frob}} : \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z}$
p prime

$$x \longmapsto x^p$$

$$(x+y)^p = x^p + \binom{p}{1} x^{p-1} y + \binom{p}{2} x^{p-2} y^2 + \cdots + \binom{p}{p-1} x y^{p-1} + y^p$$

and $p \mid \binom{p}{k}$ for $1 \le k \le p-1$

$$\Longrightarrow (x+y)^p = x^p + y^p$$

$$(\lambda x)^p = \lambda^p x^p$$

$$\text{Frob} : \mathbb{Z}/p\mathbb{Z}[x] \longrightarrow \mathbb{Z}/p\mathbb{Z}[x]$$
$$f \longmapsto f^p$$

is a lin trans

FlT:

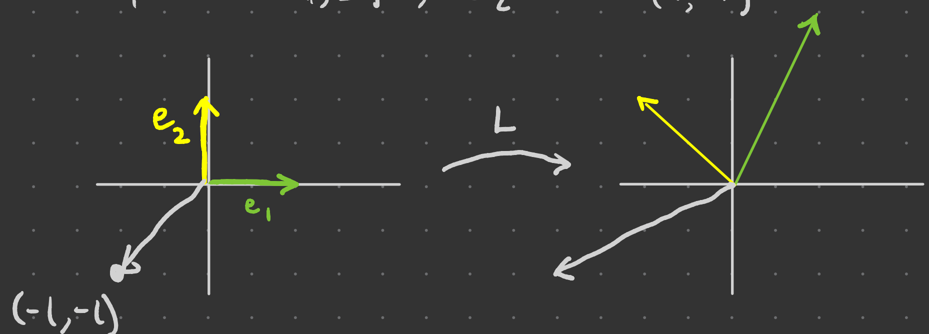
$$a^p \equiv a \pmod p$$

Aha! Frob = id

Thm (A linear transformation is determined by its action on a basis.)

V, W F-vs's, $\mathcal{B}$ a basis of V. For each $b \in \mathcal{B}$, suppose $w_b \in W$. Then $\exists!$ linear transformation $L : V \to W$ s.t. $L(b) = w_b$ for all $b \in \mathcal{B}$.

E.g. There is a unique linear trans'n $\mathbb{R}^2 \xrightarrow{L} \mathbb{R}^2$ taking $e_1 \longmapsto (1,2)$, $e_2 \longmapsto (1,-1)$:



$L(-1,-1) =$

$L(-e_1 - e_2)$

$= -L(e_1) - L(e_2)$

$= -(1,2) - (1,-1) = (-2,-1)$

Pf of Thm For $v \in V$, $\exists!\, b_1, \dots, b_n \in B$, $\lambda_1, \dots, \lambda_n \in F \setminus \{0\}$ s.t. $v = \sum_{i=1}^{n} \lambda_i b_i$. Define

$$L(v) = \sum_{i=1}^{n} \lambda_i w_{b_i}.$$

$$L(v) = L\left(\sum \lambda_i b_i\right) = \sum \lambda_i L(b_i) = \sum \lambda_i w_{b_i}$$

This is well-defined by uniqueness of b_i, λ_i, and no other expression will work b/c L needs to be linear. $\square$

🗣 "L is defined on B and then extended linearly."

In above example $e_1 \longmapsto (1,2)$, $e_2 \longmapsto (1,-1)$,

$$L(x,y) = L(xe_1 + ye_2) = xL(e_1) + yL(e_2)$$
$$= x(1,2) + y(1,-1)$$
$$= (x+y, 2x-y).$$

Defn For V, W both F-vs's, let

$$\mathrm{Hom}(V, W) := \{ L : V \to W \mid L \text{ linear} \}.$$

(Alt notation: $\mathrm{Hom}_F(V, W)$, $\mathcal{L}(V, W)$.)

This is an F-vs via

$$\begin{aligned} f + \lambda g : V &\longrightarrow W \\ v &\longmapsto f(v) + \lambda g(v) \end{aligned}$$

for $f, g \in \mathrm{Hom}(V, W)$, $\lambda \in F$.

If $W = F$, write $V^* := \underbrace{\mathrm{Hom}(V, F)}_{\text{linear functionals}}$ and call this the **dual** of V.