24. IX. 18

Goals
- define bases
- preservation of linear structure (isomorphism)

Defn A subset $B \subseteq V$ is a basis of V when
- B generates V $(\operatorname{span} B = V)$, &
- B is linearly independent

If $B \subseteq V$ is lin ind, then B is a basis for span B

Fact $B \subseteq V$ is a basis iff
- B is a minimal (wrt $\subseteq$) generating set of V

iff
- B is a maximal (wrt $\subseteq$) lin ind set.

Defn An ordered basis is a basis listed as a sequence: $b_1, b_2, \ldots$.

Note If $B = (b_1, \ldots, b_n)$ is an ordered basis of V and $v \in V$, then $\exists!$ $\lambda_1, \ldots, \lambda_n \in F$ s.t. $v = \lambda_1 b_1 + \cdots + \lambda_n b_n$.

there exists unique

Defn In the above situation, the coordinates of v wrt B are $\mathrm{Rep}_B(v) := (\lambda_1, \ldots, \lambda_n) \in F^n$

E.g. (1) F^3 has ordered basis $(e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1))$, its standard ordered basis. Since $(x,y,z) = xe_1 + ye_2 + ze_3$, the coordinates of (x,y,z) are (x,y,z). :)

(2) Take $B' = (e_3, e_2, e_1)$. Then $\mathrm{Rep}_{B'}(x,y,z) = (z,y,x)$

— order matters!

(3) One may check that $B'' = ((1,0,0), (1,1,0), (1,1,1))$ is an ordered basis of F^3. Since

$$(x,y,z) = (x-y)(1,0,0) + (y-z)(1,1,0) + z(1,1,1),$$

have $\mathrm{Rep}_{B''}(x,y,z) = (x-y,\ y-z,\ z)$. For instance,

$$\mathrm{Rep}_{B''}(1,0,3) = (1,-3,3).$$

(4) Problem Find a basis for $F^{2\times 2} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ a,b,c,d \in F \right\}$.

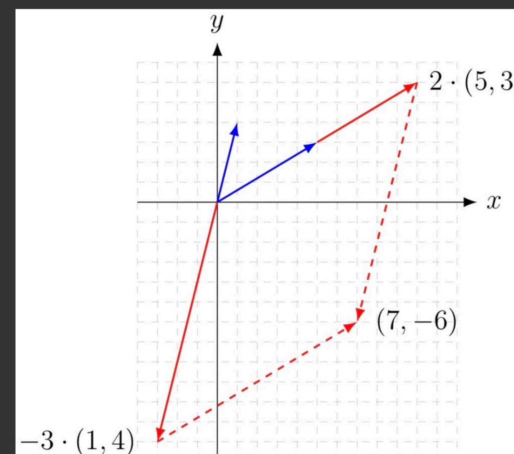
e.g. $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$

(5) Suppose we know $C = ((5,3),(1,4))$ is an ordered basis of $\mathbb{R}^2$. Let's determine $\mathrm{Rep}_C(7,-6)$:

Need $(\lambda,\mu) \in \mathbb{R}^2$ s.t. $\lambda(5,3) + \mu(1,4) = (7,-6)$, i.e.

$$\begin{aligned} 5\lambda + \mu &= 7 \\ 3\lambda + 4\mu &= -6 \end{aligned} \rightsquigarrow \left(\begin{array}{cc|c} 5 & 1 & 7 \\ 3 & 4 & -6 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right)$$

so $\mathrm{Rep}_C(7,-6) = (2,-3)$.



$x_1 - x_5 = 200$

$x_1 + x_2 = 50$

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ 1 & 0 & 0 & 0 & -1 & 200 \\ 1 & 1 & 0 & 0 & 0 & 50 \end{array}$$

Given an ordered basis $B=(v_1,\dots,v_n)$ of V, we get inverse bijections

$$V \overset{\mathrm{Rep}_B}{\underset{}{\rightleftarrows}} F^n$$

$$\lambda_1 v_1 + \cdots + \lambda_n v_n = v \longmapsto \mathrm{Rep}_B(v) = (\lambda_1, \dots, \lambda_n)$$

$$x_1 v_1 + \cdots + x_n v_n \longleftarrow\!\!\mid (x_1, \dots, x_n)$$

But these aren't *just* bijections! They also *preserve linear structure*: $f: V \longrightarrow W$ such that

V, W F-vs

- $f(u+v) = f(u) + f(v)$
- $f(\lambda u) = \lambda f(u)$

for all $u, v \in V$, $\lambda \in F$.

Such a function is called a <u>linear</u> <u>transformation</u>.

Diagrammatically:

$$\begin{tikzcd}
(u,v) \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & u+v \arrow[ddd, mapsto] \\
& V \times V \arrow[r, "+"] \arrow[d, "f \times f"'] & V \arrow[d, "f"] & \\
& W \times W \arrow[r, "+"'] & W & \\
(f(u), f(v)) \arrow[rrr, mapsto] & & & f(u+v) = f(u)+f(v)
\end{tikzcd}
\qquad
\begin{tikzcd}
(\lambda, u) \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & \lambda u \arrow[ddd, mapsto] \\
& F \times V \arrow[r, "\cdot"] \arrow[d, "id \times f"'] & V \arrow[d, "f"] & \\
& F \times W \arrow[r, "\cdot"'] & W & \\
(\lambda, f(u)) \arrow[rrr, mapsto] & & & f(\lambda u) = \lambda f(u)
\end{tikzcd}$$

both commute.

V, W F-vector spaces.

<u>Defn</u> A linear transformation $f: V \longrightarrow W$ is called an <u>isomorphism</u> when it admits a two-sided inverse

$g: W \longrightarrow V$ that is also a linear transformation.

In this case, call V, W isomorphic and write $V \cong W$.

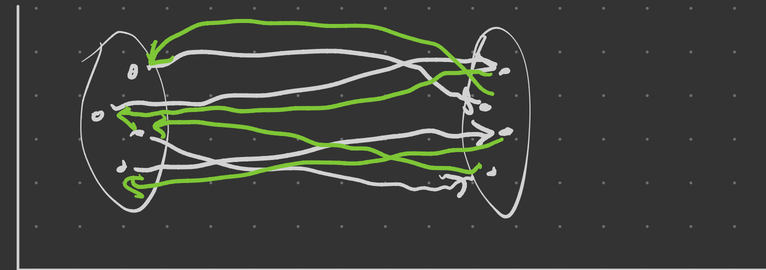
E.g. $\text{Rep}_B : V \xrightarrow{\cong} F^n$ for B any ordered basis of V.
with n elements

only one
n will work
dimension theory

Prop A function $f: V \longrightarrow W$ is an isomorphism iff it is a bijective linear transformation.

Pf ($\Rightarrow$) Assume f is an iso. Then f admits an inverse lin trans'n. In particular, that's a 2-sided inverse to f as a function. By Math 112, f is bijective.

($\Leftarrow$) Assume $f: V \to W$ is a linear bijection.



Let $g = f^{-1}$ (as a function). Want to show g is linear.

~~Note~~ $\cancel{g(u+\lambda v) = g(f}$

Take $u, v \in W$, $\lambda \in F$. Since f is bij, $\exists\, u', v' \in V$ s.t. $f(u') = u$, $f(v') = v$. Thus $g(u+\lambda v) = g(f(u') + \lambda f(v'))$

$= g(f(u' + \lambda v'))$ since f is linear.

$= u' + \lambda v'$ since $g \circ f = id_V$

$= g(u) + \lambda g(v)$ (inverses). $\square$