24. IX. 20

Goals
- finite dimension is well-defined
- learn to compute $\dim V$

Defn A vector space is finite dimensional when it has a basis with finitely many elements.

E.g.
- F^n, $F^{m\times n}$ have bases of cardinality $n, mn < \infty$, resp.
- $F[x]$, $\mathbb{R}^{\mathbb{R}}$ are infinite dimensional

Defn If V is a finite dimensional F-vs, then the dimension of V, denoted $\dim V = \dim_F V$, is the cardinality of any basis of V.

2 Is this well-defined?

Exchange lemma Suppose $B=\{v_1,\dots,v_n\}$ is a basis of V and

$$w \overset{\star}{=} \sum_{i=1}^{n} \lambda_i v_i$$ with $\lambda_i \in F$ not all 0. If $\lambda_\ell \neq 0$ for some $\ell \in \{1,\dots,n\}$,

then $B' = \underbrace{(B \setminus \{v_\ell\}) \cup \{w\}}_{v_\ell,\ w \text{ exchanged}}$ is also a basis of V.

Thus $B' = \{w, v_2, v_3, \dots, v_n\}$

Pf First show B' lin ind. WLOG, $\ell = 1$. Suppose $\mu w + \mu_2 v_2 + \cdots + \mu_n v_n = 0$.

Subbing $\star$, $\mu\left(\sum_{i=1}^{n} \lambda_i v_i\right) + \mu_2 v_2 + \cdots + \mu_n v_n = 0$

$$\iff \mu\lambda_1 v_1 + (\mu\lambda_2 + \mu_2)v_2 + \cdots + (\mu\lambda_n + \mu_n)v_n = 0.$$

Since B lin ind, $\mu\lambda_1 = \mu\lambda_2 + \mu_2 = \cdots = \mu\lambda_n + \mu_n = 0$.

Since $\lambda_1 \neq 0$, know $\mu = 0$, whence $\mu_2 = \cdots = \mu_n = 0$.

Thus $B' = \{w, v_2, \ldots, v_n\}$ is lin ind.

$w = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n$

(still with $l = 1$, $\lambda_1 \neq 0$)

Now show $\operatorname{span} B' = V$. Solving for v_1 in ☆ gives

$$v_1 = \frac{1}{\lambda_1} w - \frac{\lambda_2}{\lambda_1} v_2 - \cdots - \frac{\lambda_n}{\lambda_1} v_n .$$ ☆

For $v \in V$, B a basis $\Rightarrow$

$$v = \mu_1 v_1 + \cdots + \mu_n v_n \text{ for some } \mu_i \in F.$$

Subbing in ☆ gives

$$v = \mu_1\left(\frac{1}{\lambda_1} w - \frac{\lambda_2}{\lambda_1} v_2 - \cdots - \frac{\lambda_n}{\lambda_1} v_n\right) + \mu_2 v_2 + \cdots + \mu_n v_n$$

$$= \frac{\mu_1}{\lambda_1} w + \left(\mu_2 - \frac{\mu_1 \lambda_2}{\lambda_1}\right) v_2 + \cdots + \left(\mu_n - \frac{\mu_1 \lambda_n}{\lambda_1}\right) v_n$$

$\in \operatorname{span} B'$.

Thus span $B' = V$ and we've already seen B' lin ind, so B' is a basis. $\square$

Thm In a finite dimensional vector space V, every basis has the same cardinality.

Pf Among all bases of V, let $B = \{v_1, \dots, v_n\}$ be one of minimal cardinality. Let $C = \{w_1, w_2, \dots\}$ be another basis of V.

WTS: $|C| = |B|$. Know $n = |B| \leq |C|$.

Idea: Use the exchange lemma to swap n elts of C into B while maintaining basis status.

Let $B_0 = B$, take $w_1 \in C$. By the exchange lemma,

get new basis B_1 by swapping w_1 in for some $v_l \in B_0$.

WLOG, $l = 1$ and $B_1 = \{w_1, v_2, \dots, v_n\}$ is a basis of V.

Take $w_2 \in C \setminus \{w_1\}$.

Since B_1 is a basis,

$$w_2 = \lambda_1 w_1 + \lambda_2 v_2 + \dots + \lambda_n v_n \quad \text{for some } \lambda_i \in F.$$

Since w_1, w_2 are lin ind, some λ_l, $l \geq 2$ is nonzero.

WLOG, $l = 2$ and we can exchange to get $B_2 = \{w_1, w_2, v_3, \dots, v_n\}$

a basis of V.

Continuing in this fashion, eventually get

$B_n = \{w_1, \dots, w_n\}$ is a basis of V

$\cap$

C In fact, $B_n = C$ b/c o/w $w_{n+1} \in \operatorname{span} B_n$

because otherwise

$\Rightarrow C$ lin dep. ※ $\square$

<u>Cor</u> dimension of fin dim vs's is well-defined

<u>Cor</u> If V is a fin dim vs, $S \subseteq V$ lin ind, then we may extend S to a basis of V by adding some $\dim V - |S|$ elements.

<u>Pf</u> Apply the "basis production algorithm" from the theorem's proof. $\square$

<u>Cor</u> If V fin dim vs and $T \subseteq V$ generates V, then some subset $S \subseteq T$ is a basis of V. $\square$

$\hookrightarrow$ Suppose $S \subseteq V$ lin ind but span $S \subsetneq V$. Then take $w \in V \setminus \text{span } S$. <u>Claim</u> $S \cup \{w\}$ is lin ind.

this is the case iff $\forall v \in S \cup \{w\}$, v is <u>not</u> a lin combo of elts of $(S \cup \{w\}) \setminus \{v\}$.

True since S lin ind + $w \in V \setminus \text{span } S$. $\square$

#6 $\quad S = \{(a,b), (c,d)\}$ lin ind?

$$\Leftrightarrow \left(\begin{array}{cc|c} a & c & 0 \\ b & d & 0 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} \underline{1} & \underline{0} & 0 \\ \underline{0} & \underline{1} & 0 \end{array}\right)$$

justify

"ad-bc" $= 1\cdot 1 - 0 \cdot 0$
$= 1$

- row ops
- what happens to ad-bc when you apply a row op?