24. IX. 11

Goals

- Two operations & eight properties of vector spaces
- Become familiar with standard and nonstandard examples of vector spaces
- Define subspaces

Fix a field F.

Defn A *vector space* over F is a set V together with operations

$$+ : V \times V \longrightarrow V \qquad \text{vector addition}$$

$$\cdot : F \times V \longrightarrow V \qquad \text{scalar multiplication}$$

such that for all $u, v, w \in V$, $\lambda, \mu \in F$

(1) $u + v = v + u$ commutativity of +

(2) $u + (v + w) = (u+v) + w$ associativity of +

(3) $\exists\, 0 \in V$ s.t. $0 + u = u$ additive identity

(4) $\exists\, -u \in V$ s.t. $u + (-u) = 0$ additive inverse

(5) $F \ni 1 \cdot u = u$ scalar multiplicative identity

(6) $\lambda \cdot (\mu \cdot v) = (\underbrace{\lambda \mu}_{\text{product in } F}) \cdot v$ associativity of scalar mult'n

(7) $\lambda \cdot (u + v) = (\lambda \cdot u) + (\lambda \cdot v)$ distribution of scalar mult'n over vector add'n

(8) $(\lambda + \mu) \cdot u = (\lambda \cdot u) + (\mu \cdot u)$ distribution of scalar mult'n over field add'n

<u>Note</u> (1) - (4) make $(V, +)$ an Abelian group

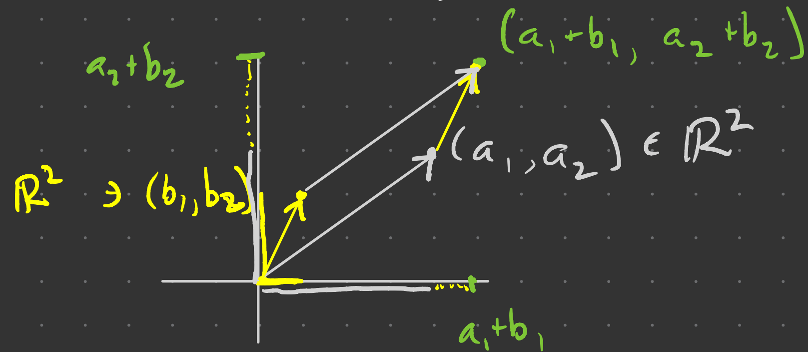
Cannot, in gen'l, multiply vectors!

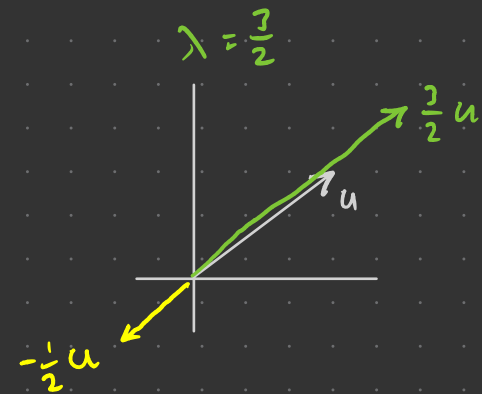
E.g. • For $n \in \mathbb{N}$, $F^n = \underbrace{F \times \cdots \times F}_{n \text{ times}} = \{(a_1, a_2, \dots, a_n) \mid a_i \in F, 1 \le i \le n\}$

with $(a_1, \dots, a_n) + (b_1, \dots, b_n) = (a_1 + b_1, \dots, a_n + b_n)$

and $\lambda \cdot (a_1, \dots, a_n) = (\lambda a_1, \dots, \lambda a_n)$

• For $n = 2$, $F = \mathbb{R}$, get Euclidean plane:





• $F = \mathbb{Z}/3\mathbb{Z}$, $n = 4$: $\quad (1,0,2,1) + 2(0,0,1,1) = (1,0,2,1) + (0,0,2,2)$

$$= (1,0,1,0) \in F^4$$

- $\mathbb{C}$ is an $\mathbb{R}$-vs via $\lambda(a+bi) = (\lambda a) + (\lambda b)i$, $a, b, \lambda \in \mathbb{R}$

- If $F \subseteq L$ are both fields, then L is an F-vs, e.g. $\mathbb{R}$ is a $\mathbb{Q}$-vs.

- $F^{m \times n} := \{ m \times n$ matrices with entries in $F \}$

$$= \left\{ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \middle| \; a_{ij} \in F, \; 1 \leq i \leq m, \; 1 \leq j \leq n \right\}$$

For $A \in F^{m \times n}$, write A_{ij} for its ij-th entry. Then for $A, B \in F^{m \times n}$, $\lambda \in F$

$$(A+B)_{ij} := A_{ij} + B_{ij}$$

$$(\lambda A)_{ij} := \lambda A_{ij}$$

So for $F = \mathbb{Q}$, $m = 2$, $n = 3$,

$$2\begin{pmatrix} 1 & 2 & 1 \\ 3 & 4 & 3 \end{pmatrix} - 5\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 2 \\ 1 & 8 & 1 \end{pmatrix}$$

- If S is any set, then $F^S := \{ f \mid f : S \to F \text{ a function} \}$ is an F-vs where for $f, g \in F^S$, $\lambda \in F$:

$$f+g : S \longrightarrow F$$
$$s \longmapsto f(s) + g(s)$$

$$\lambda \cdot f : S \longrightarrow F$$

$$s \longmapsto \lambda f(s)$$

Note For $n \in \mathbb{N}$, let $\underline{n} := \{1, 2, \dots, n\}$. Then

$$F^{\underline{n}} \;\text{“}\overset{\text{isomorphic } \cong}{=}\text{”}\; F^{n}$$

$$f \longmapsto (f(1), f(2), \dots, f(n))$$

$$\begin{array}{c} \underline{n} \\ \downarrow \\ F \end{array} \quad \begin{array}{c} i \\ \downarrow \\ a_i \end{array} \longleftarrow (a_1, \dots, a_n)$$

$$F^{\underline{m} \times \underline{n}} \;\text{“}=\text{”}\; F^{m \times n}$$

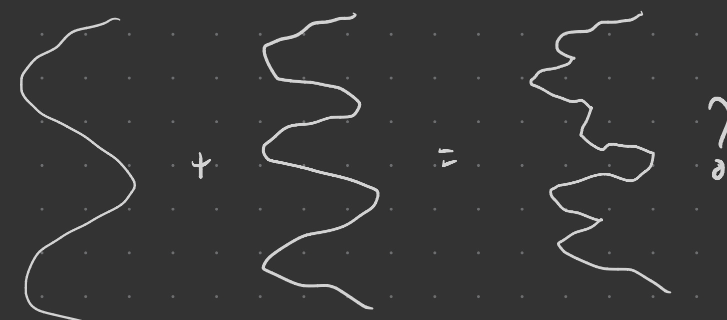
$$\underline{m} \times \underline{n} = \{(i,j) \mid i \in \underline{m}, j \in \underline{n}\}$$

$$f \longmapsto (f(i,j))_{ij}$$

$$\begin{array}{c} (i,j) \\ \downarrow \\ A_{ij} \end{array} \longleftarrow A$$

Note $\mathbb{R}^{\mathbb{R}} = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$ is a vector space that calculus students love.

Q Other examples?



Subspaces

$$\mathbb{R}^{\mathbb{R}}$$
$$\cup\!|$$
$$C(\mathbb{R}) = \{ f \mid f : \mathbb{R} \to \mathbb{R} \text{ is continuous} \}$$
$$\cup\!|$$
$$C'(\mathbb{R}) = \{ f \mid f : \mathbb{R} \to \mathbb{R} \text{ with } f' \text{ continuous} \}$$
$$\cup\!|$$
$$C^\infty(\mathbb{R}) = \{ f \mid f : \mathbb{R} \to \mathbb{R} \text{ has continuous derivatives of all orders} \}$$
$$\cup\!|$$
$$\mathbb{R}[x] = \{ f \mid f : \mathbb{R} \to \mathbb{R} \text{ is a polynomial function} \}$$

Each of these satisfies $f, g \in W \Rightarrow f+g \in W$, $\lambda f \in W$, $0 \in W$

In fact, each is a vector space in its own right.