24. IX. 9

Goals
- Define echelon & reduced echelon forms
- Every system has a unique reduced echelon form (REF) (w/ algorithm)
- Solution sets from REF.

leave out augmentation

Defn A matrix is in <u>echelon form</u> when

(1) all rows of just 0's are at the bottom

(2) the <u>pivot</u> of each nonzero row is strictly to the right of the pivot of the previous row.

first nonzero term in the row

E.g.

$$\begin{pmatrix} 1 & * & * & * & * \\ 0 & 0 & 2 & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Prop We can put any matrix in echelon form via row ops of type (2) & (3).

(2) swap; (3) add λr_j to r_i, $\lambda \in F$

Pf (by algorithm)

(1) Swap rows to get (any) leftmost pivot in top row

(2) Use (3) to get 0's below this pivot

(3) Repeat previous steps with the matrix obtained by deleting first row & column. $\square$

E.g. System:

$$\begin{array}{rcrcrcrcrcl}
 & & & & 2x_3 & + & 6x_4 & & & = & 0 \\
x_1 & + & 2x_2 & + & x_3 & + & 3x_4 & & & = & 1 \\
2x_1 & + & 4x_2 & + & 3x_3 & + & 9x_4 & + & x_5 & = & 5
\end{array}$$

Augmented matrix:

$$\left(\begin{array}{ccccc|c} 0 & 0 & 2 & 6 & 0 & 0 \\ 1 & 2 & 1 & 3 & 0 & 1 \\ 2 & 4 & 3 & 9 & 1 & 5 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 3 & 0 & 1 \\ 0 & 0 & 2 & 6 & 0 & 0 \\ 2 & 4 & 3 & 9 & 1 & 5 \end{array}\right)$$

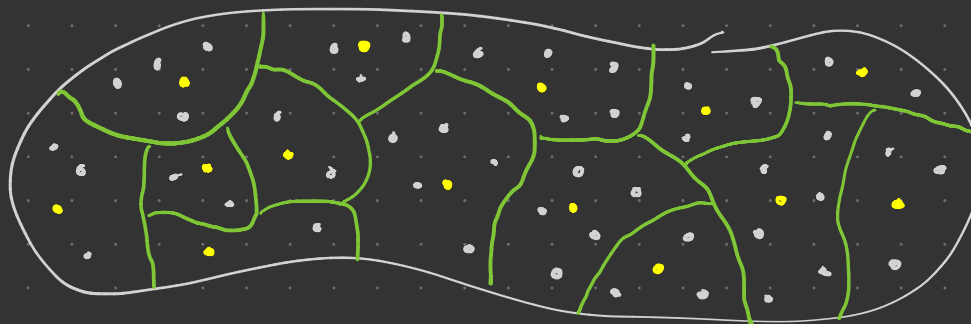
$$\xrightarrow{r_3 \to r_3 - 2r_1} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 3 & 0 & 1 \\ 0 & 0 & 2 & 6 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 3 \end{array}\right) \xrightarrow{r_3 \to r_3 - \frac{1}{2} r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 3 & 0 & 1 \\ 0 & 0 & 2 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right)$$

Now in echelon form!

REF

Defn A matrix is in <u>reduced echelon form</u> when it is in echelon form, each pivot is 1, and above and below each pivot the column is all 0's.

Thm Every matrix is equivalent (via row operations) to a unique matrix in REF.



$F^{m \times n}$

- not in REF
- in REF

equivalence classes

pf To produce a REF representative, first pass to echelon form, then scale rows by $\frac{1}{p_i}$ where p_i = pivot of i^{th} row, then use

(3) to clear the column above and below each pivot.

Uniqueness: One. III. 2. 6. $\square$

E.g.

$$\left(\begin{array}{ccccc|c} 1 & 2 & 1 & 3 & 0 & 1 \\ 0 & 0 & 2 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow{r_2 \to \frac{1}{2} r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{r_1 \to r_1 - r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right)$$

pivots $\longleftrightarrow$ pivot variables x_1, x_3, x_5

Other variables are free

Solution set $\left\{ (x_1, \dots, x_5) \in F^5 \;\middle|\; \begin{aligned} x_1 + 2x_2 &= 1 \\ x_3 + 3x_4 &= 0 \\ x_5 &= 3 \end{aligned} \right\}$

Solving for pivots: $\quad x_1 = 1 - 2x_2$

$\qquad\qquad\qquad\quad\; x_3 = -3x_4$

$\qquad\qquad\qquad\quad\; x_5 = 3$

This gives the parametric form of the solution set

$$\left\{ (\underline{1-2x_2}, x_2, \underline{-3x_4}, x_4, \underline{3}) \;\middle|\; x_2, x_4 \in F \right\}.$$

pivots (now in terms of free) $\qquad$ free

and vector form

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix} \;\middle|\; x_2, x_4 \in F \right\}.$$

Problem Find all parabolas $f(x) = ax^2 + bx + c$ passing through $(1,4)$ and $(3,6)$.

Solution We know
$$4 = a+b+c$$
$$6 = 9a+3b+c$$
resulting in the augmented matrix $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 9 & 3 & 1 & 6 \end{array}\right)$. We now apply Gaussian reduction:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 9 & 3 & 1 & 6 \end{array}\right) \xrightarrow{r_2 \to r_2 - 9r_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -6 & -8 & -30 \end{array}\right)$$

$$\xrightarrow{r_2 \to -\frac{1}{6} r_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{4}{3} & 5 \end{array}\right) \xrightarrow{r_1 \to r_1 - r_2} \left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & -1 \\ 0 & 1 & \frac{4}{3} & 5 \end{array}\right)$$

$$\left\{ \left(\frac{1}{3}c - 1,\ -\frac{4}{3}c + 5,\ c \right) \,\middle|\, c \in \mathbb{R} \right\}$$