24. IX. 6

Goals
- Algebra & geometry of systems of linear eq'ns
- Intro to Gaussian elimination / row reduction

Note Today, $F = \mathbb{R}$. Non-geometry work over any field.

Example 1 Solve
$$\left.\begin{aligned} 3x + 2y &= 5 \\ 2x - y &= 1. \end{aligned}\right\} \text{ system of linear eq'ns in 2 variables } x, y$$

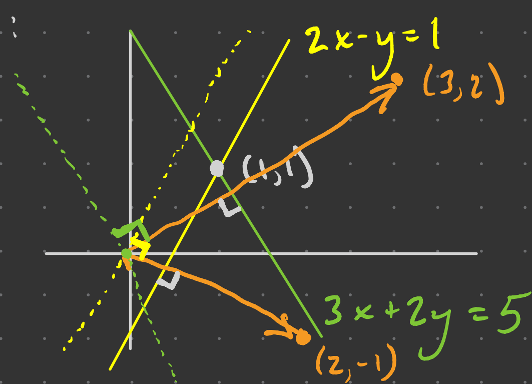
Multiply 2nd eq'n by 2, then add eq'ns to eliminate y:
$$\begin{array}{rrcl} & 3x + 2y &=& 5 \\ + & 4x - 2y &=& 2 \\ \hline & 7x &=& 7 \end{array} \implies x = 1$$

Sub $x=1$ onto first eq'n to get

$$3\cdot 1+2y=5 \Rightarrow y=1.$$

Thus $x=y=1$ is the unique solution.

Geometrically:



Question What is the relationship b/w $2x-y=1$ and $(2,-1)$?

$3x+2y=5$ and $(3,2)$?

Example 2 System $-9x - 3y = 6$ (1)

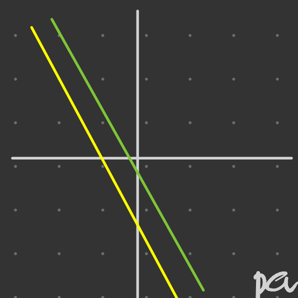
$3x + y = -2$ (2)

Here (1) $= -3 \cdot$ (2) so they have the same solutions.

Example 3 System $-9x - 3y = 6 \xrightarrow{\cdot \frac{-1}{3}} 3x + y = -2$

$3x + y = -1$

Cannot both be true!
Thus no solutions.



parallel lines — same normal vectors!

Gaussian elimination

Idea: Transform a system into a new one with

(a) same set of solutions

(b) evident solutions

\lambda

Legal transformations:

(1) Multiply an eq'n by $\lambda \in F^\times = \{ x \in F \mid x \neq 0 \}$

(2) Swap two eq'ns

(3) Add a multiple of one eq'n to another

Discuss Why are these "legal"?

Augmented matrices

$$\begin{aligned} x + 2y + z &= 0 \\ x \quad\quad + z &= 4 \\ x + y + 2z &= 1 \end{aligned} \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 2 & 1 \end{array}\right)$$

Just record the coefficients!

Notation: r_i = i-th row of augmented matrix

Now eliminate:

elim x from 2nd eqn

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 2 & 1 \end{array}\right) \xrightarrow[r_3 \to r_3 - r_1]{r_2 \to r_2 - r_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -2 & 0 & 4 \\ 0 & -1 & 1 & 1 \end{array}\right)$$

elim x from 3rd eqn

$$\xrightarrow[\substack{\text{solve for } y \\ \text{in 2nd eqn}}]{r_2 \to -\frac{1}{2} r_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & -1 & 1 & 1 \end{array}\right) \xrightarrow[\substack{\text{elim } y \\ \text{from 3rd eqn}}]{r_3 \to r_3 + r_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\xrightarrow[\substack{\text{elim } y \\ \text{from 1st eqn}}]{r_1 \to -2r_2 + r_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow{r_1 \to -r_3 + r_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right).$$

Thus the unique sol'n is $(x, y, z) = (5, -2, -1)$.

<u>Question</u> Check this!

Example 4 Solve

$$\begin{aligned} x + 2y + z &= 0 \\ x + z &= 4 \\ x + y + z &= 1 . \end{aligned}$$

Then

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 1 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

and thus $x = 4$, $y = -2$, $0 = -1$

There are <u>no</u> sol'ns to this system.

Example 5 Another small modification:

$$\begin{aligned} x + 2y + z &= 0 \\ x + z &= 4 \\ x + y + z &= 2 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 1 & 2 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So
$$\begin{aligned} x + z &= 4 \\ y &= -2 \\ 0 &= 0 \end{aligned} \implies \text{sol'n set } \{(x, -2, 4-x) \mid x \in \mathbb{R}\}$$

a line in $\mathbb{R}^3$.

Note Each "row reduction" ended in "reduced echelon form".

To do: (a) define this

(b) prove Gaussian reduction always gives this form

(c) understand sol'n sets from REF.