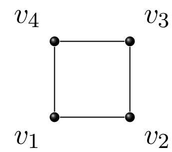
MATH 201: LINEAR ALGEBRA
HOMEWORK DUE FRIDAY WEEK 11

Problem 1. Consider the *cycle graph* C_4:



(a) Find the adjacency matrix $A = A(G)$.
(b) Compute A^4 and use it to determine the number of walks from v_1 to v_3 of length 4. List all of these walks (these will be ordered lists of 5 vertices).
(c) What is the total number of *closed* walks of length 4?
(d) Compute and factor the characteristic polynomial for A.
(e) What are the algebraic multiplicities of each of the eigenvalues?
(f) Diagonalize A using our algorithm: compute bases for the eigenspaces of each of the eigenvalues you just found, and use them to construct a matrix P such that $P^{-1}AP$ is a diagonal matrix with the eigenvalues along the diagonal.
(g) Use part (f) to find a closed expression for A^ℓ for each $\ell \geq 1$. Use this expression to then give separate expressions for the cases where ℓ is even and where ℓ is odd.
(h) Use part (g) to take the trace of A^ℓ to get a formula for the number of closed walks of length ℓ for each $\ell \geq 1$.

Problem 2. In this exercise we will prove the theorem from class:

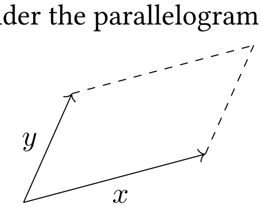
> Let A be the adjacency matrix for a graph G with vertices $v_1, \ldots, v_n$, and let $\ell \in \mathbb{N}$. Then then number of walks of length ℓ from v_i to v_j is $(A^\ell)_{ij}$.

(a) Let $p(i, j, \ell)$ denote the number of walks of length ℓ in G from v_i to v_j. Prove that for all $i, j = 1, \ldots, n$ and $\ell \geq 1$,

$$p(i, j, \ell) = \sum_{k=1}^{n} p(i, k, \ell - 1)p(k, j, 1).$$

(*Hint:* Part of the trick is to parse this formula appropriately.)
(b) Prove the theorem by induction on ℓ, using the result from part (a).

Problem 3. A *rhombus* is a parallelogram with all sides of equal length. Using the standard inner product in $\mathbb{R}^2$, prove that a parallelogram is a rhombus if and only if its diagonals meet perpendicularly. Hint: take two arbitrary vectors $x, y \in \mathbb{R}^2$ and consider the parallelogram determined by x and y:



(Another hint: avoid coordinates. I.e., you should use abstract properties of the inner product and definitions of length and angle rather than formulas involving x_1, x_2, y_1, y_2.)

Problem 4. Let V be an n-dimensional vector space over $F = \mathbb{R}$ or $\mathbb{C}$, and let $\langle\, , \,\rangle$ be an inner product on V. Let $\alpha = \{v_1, \ldots, v_n\}$ be an ordered basis for V (not necessarily orthogonal). Let A be the $n \times n$ matrix given by

$$A_{ij} = \langle v_i, v_j \rangle.$$

For $x \in V$, let $[x]_\alpha \in F^n$ denote the coordinate vector for x with respect to the basis α. So if $x = \sum_{i=1}^n a_i v_i$, then $[x]_\alpha = (a_1, \ldots, a_n)$. (We have denoted this $\mathrm{Rep}_\alpha(x)$ in the past.) As usual, we will think of this vector in F^n as an $n \times 1$ matrix.

(a) Prove that for all $x, y \in V$,

$$\langle x, y \rangle = ([x]_\alpha)^\top A \left(\overline{[y]_\alpha}\right).$$

(Recall that for a matrix C, we define $\overline{C}$ by $\overline{C}_{ij} = \overline{C_{ij}}$, and then we define the conjugate transpose by $C^* = \overline{C^\top}$. *Hint*: compute both sides using sum notation. On the right-hand side, you will be computing the $1, 1$-entry of a 1×1 matrix.)

(b) Prove that the matrix A satisfies $A = A^*$.

(c) If the basis α is orthonormal, what is the matrix A?

(d) (Extra credit) Let $\mathcal{D}$ be another ordered basis for V, and let C be the associated $n \times n$ matrix. How are A and C related (with proof)?

Problem 5. Let V be the vector space of all continuous functions $[0, 1] \to \mathbb{R}$ with inner product $\langle f, g \rangle = \int_0^1 f(t)g(t)\, dt$. Let W be the subspace spanned by $\left\{t, \sqrt{t}\right\}$. (Warning: to get this problem right, you will need to be very careful with your calculations and double-check your solutions.)

(a) Apply Gram-Schmidt to $\left\{t, \sqrt{t}\right\}$ to compute an orthonormal basis $\{u_1, u_2\}$ for W. (Hint: the coefficient of t in u_2 should be $-6\sqrt{2}$.)

(b) Find the closest function in W to $f(t) = t^2$. Express your solution in two forms: (i) as a linear combination of u_1 and u_2, and (ii) as a linear combination of t and $\sqrt{t}$.

(c) Graph f and its projection onto W (which you just calculated).