# MATH 201: LINEAR ALGEBRA
# HOMEWORK DUE FRIDAY WEEK 9

*Problem* 1. For each of the following matrices $A \in M_{n\times n}(F)$

(i) Determine all eigenvalues of $A$.
(ii) For each eigenvalue $\lambda$ of $A$, find the set of eigenvectors corresponding to $\lambda$.
(iii) If possible, find a basis for $F^n$ consisting of eigenvectors of $A$.
(iv) If successful in finding such a basis, determine an invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$.

(a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ for $F = \mathbb{R}$.

(b) $A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}$ for $F = \mathbb{R}$.

(c) $A = \begin{pmatrix} 7 & -5 \\ 10 & -7 \end{pmatrix}$ for $F = \mathbb{R}$.

(d) $A = \begin{pmatrix} 7 & -5 \\ 10 & -7 \end{pmatrix}$ for $F = \mathbb{C}$.

(e) $A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix}$ for $F = \mathbb{R}$.

*Problem* 2. Let $V = \mathbb{R}[x]_{\leq 3}$, the vector space of polynomials of degree at most 3 with real coefficients. Let $L$ denote the linear endomorphism

$$\begin{aligned} L\colon V &\longrightarrow V \\ p(x) &\longmapsto xp'(x) + p'(x). \end{aligned}$$

(You do not need to prove that $L$ is linear, but you should know how to.) Find the eigenvalues of $L$ and determine if $V$ has a basis of eigenvectors of $L$. If $L$ has such a basis, provide one (written as a set of polynomials), and if not, explain why not.

*Problem* 3. Let $f\colon V \to V$ be a linear transformation. For a positive integer $m$, we define $f^m$ inductively as $f \circ f^{m-1}$. Prove that if $\lambda$ is an eigenvalue for $f$, then $\lambda^m$ is an eigenvalue for $f^m$.

*Problem* 4. Define $T\colon \mathsf{Mat}_{n\times n}(\mathbb{R}) \to \mathsf{Mat}_{n\times n}(\mathbb{R})$ by $T(A) = A^\top$ (the transpose of $A$).

(a) Show that the only eigenvalues of $T$ are 1 and -1. (*Hint:* Problem 3 might help.)
(b) For $n = 2$, describe the eigenvectors corresponding to each eigenvalue.
(c) Find an ordered basis $\alpha$ for $\mathsf{Mat}_{2\times 2}(\mathbb{R})$ such that the matrix that represents $T$ with respect to $\alpha$ is diagonal.
(d) Repeat part (b) for an arbitrary $n > 2$.
(e) Repeat part (c) for $\mathsf{Mat}_{n\times n}(\mathbb{R})$.