MATH 201: LINEAR ALGEBRA
HOMEWORK DUE FRIDAY WEEK 8

Problem 1. Let

$$A = \begin{pmatrix} 1 & -2 & 1 & 2 \\ 2 & -4 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}.$$

(a) Find elementary matrices $E_1, \dots, E_\ell$ such that $E_\ell \cdots E_2 E_1 A$ is the reduced echelon form of A. (Check your work.)
(b) Compute $\det A$ via row operations.
(c) Compute $\det A$ via permutation expansion.[1]
(d) Compute $\det A$ via Laplace expansion (along a row or column of your choosing).

Problem 2. In this exercise, we will prove that the determinant is multiplicative, that is, that for $n \times n$ matrices A and B,

$$\det(AB) = \det(A)\det(B).$$

We break the problem into two parts, depending on whether $\det(B)$ is zero or nonzero.

(a) First, let B be a fixed $n \times n$ matrix over F such that $\det(B) \neq 0$. Consider the function

$$d\colon M_{n\times n}(F) \longrightarrow F$$

defined by $d(A) = \det(AB)/\det(B)$. (Hint: the answer to the following set of questions should follow trivially from associativity of matrix multiplication. Make sure to mention why in your solution.)
   (i) Let E be the $n \times n$ elementary matrix obtained from the identity matrix by swapping two rows. Show that $d(EA) = -d(A)$.
   (ii) Let E be the $n \times n$ elementary matrix obtained from the identity matrix by scaling a row by a scalar λ. Show that $d(EA) = \lambda d(A)$.
   (iii) Let E be the $n \times n$ elementary matrix obtained from the identity matrix by adding a scalar multiple of one row to another. Show that $d(EA) = d(A)$.
   (iv) Show that $d(I_n) = 1$.

(b) It remains to be shown that $\det(AB) = \det(A)\det(B)$ when $\det(B) = 0$. Fix any $n \times n$ matrix B such that $\det(B) = 0$. Our goal is to show $\det(AB) = \det(A)\det(B) = 0$. Recall that the kernel (nullity) of a linear function $f\colon V \to W$ is the subspace of V defined by

$$\ker(f) := \{v \in V : f(v) = 0\},$$

and that the image (range) of f is the subspace of W defined by

$$\operatorname{im}(f) := \{f(v) : v \in V\}.$$

The *rank* of f is the dimension of $\operatorname{im}(f)$. If A is any matrix representing f (with respect to some choice of bases for V and W), then we have seen that the rank of A, defined as the dimension of its row or column space, equals the rank of f.

[1] Many of the entries in this matrix are 0. You are free to compute fewer than the standard $4! = 24$ terms in the permutation expansion as long as you clearly explain which permutations need not be included.

(i) Let $f\colon V \to W$ and $g\colon W \to U$ be linear transformations of finite dimensional vector spaces over F. Show that

$$\ker(f) \subseteq \ker(g \circ f) \qquad \text{and} \qquad \operatorname{im}(g \circ f) \subseteq \operatorname{im}(g).$$

(Recall that to show an inclusion of sets $A \subseteq B$, we start with: "Let $a \in A$", do some math, and conclude with "Thus, $a \in B$.")

(ii) Use part (a) to prove that $\operatorname{rank}(g \circ f) \leq \operatorname{rank}(f)$ and $\operatorname{rank}(g \circ f) \leq \operatorname{rank}(g)$.
(*Hint:* For one of them you might need to use the rank-nullity theorem.)

(iii) Let A be an $m \times n$ matrix over F, and B an $n \times p$ matrix over F. Use what you have already shown to prove that $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$ and $\operatorname{rank}(AB) \leq \operatorname{rank}(B)$.

(iv) Using the previous parts of this problem, prove that if A and B are $n \times n$ matrices such that either $\det(A) = 0$ or $\det(B) = 0$, then $\det(AB) = 0$.

Problem 3. Read the attached exposition on Cramer's rule before attempting this problem.

(a) Consider the 3×3 system of equations over the real numbers:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}.$$

Use Cramer's rule to compute x_2. (You may assume the system is consistent.)

(b) Consider the following matrix over the complex numbers:

$$A = \begin{pmatrix} 1+i & 0 & 0 \\ 0 & 1 & 0 \\ i & 0 & 1-i \end{pmatrix}.$$

Compute each entry of $\operatorname{adj}(A)$ by hand, and then use the formula coming from Cramer's rule to compute A^{-1}.

Let A be an invertible $n \times n$ matrix, and let $b \in F^n$. Consider the $n \times n$ system of linear equations $Ax = b$ where x is the column vector with entries $(x_1, \ldots, x_n)$. For each $j = 1, \ldots, n$, let M_j be the $n \times n$ matrix formed by replacing the j-th column of of A by b. *Cramer's rule* says that the solution to the system is given by

$$x_j = \frac{\det(M_j)}{\det(A)},$$

for $j = 1, \ldots, n$.

Example 1. Consider the system of equation

$$\begin{aligned} ax + by &= s \\ cx + dy &= t. \end{aligned}$$

In matrix form, we write the system as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s \\ t \end{pmatrix}.$$

Assume the determinant of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is nonzero, and apply Cramer's rule:

$$\det(M_1) = \det \begin{pmatrix} s & b \\ t & d \end{pmatrix} = sd - bt,$$

$$\det(M_2) = \det \begin{pmatrix} a & s \\ c & t \end{pmatrix} = at - sc.$$

By Cramer's rule, the solution to the system is

$$x = \frac{\det(M_1)}{\det(A)} = \frac{sd - bt}{ad - bc}$$

$$y = \frac{\det(M_2)}{\det(A)} = \frac{at - sc}{ad - bc}.$$

Let's check this solution:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s \\ t \end{pmatrix} \implies \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} s \\ t \end{pmatrix}.$$

It is easy to check that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Hence, the solution is

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} s \\ t \end{pmatrix} \\ &= \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} \\ &= \frac{1}{ad - bc} \begin{pmatrix} ds - bt \\ -cs + ta \end{pmatrix}. \end{aligned}$$

This agrees with the solution we calculated using Cramer's rule.

Cramer's rule and inverses. Suppose that A is an invertible $n \times n$ matrix. To find the inverse of A, we need to find a matrix X such that $AX = I_n$. Finding the j-th column of X is the same as solving the system $Ax = e_j$, where e_j is the j-th standard basis vector. We can then solve for x using Cramer's rule n times—once for each column. We now describe the resulting formula for the inverse of A. First, some notation: for each $i, j \in \{1, \ldots, n\}$ let A^{ji} be the $(n-1) \times (n-1)$ matrix formed by removing the j-th row and i-th column of A. Next, define the *adjugate* of A, denoted $\mathrm{adj}(A)$ by

$$(\mathrm{adj}(A))_{ij} = (-1)^{i+j} \det(A^{ji}).$$

The scalar $(-1)^{i+j} \det(A^{ji})$ is called the *ji-th cofactor of A*. Note that we are defining the ij-th entry of the adjugate using the ji-th cofactor—the indices reverse order.

Cramer's rule applied to the problem of finding the inverse then gives the following important formula:

$$A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A).$$

Example 2. As a simple example of Cramer's formula for the inverse, let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

In this case, each A^{ji} is a 1×1 matrix. We get

$$\begin{aligned}
(\mathrm{adj}(A))_{11} &= (-1)^{1+1} \det(A^{11}) = \det([d]) = d \\
(\mathrm{adj}(A))_{12} &= (-1)^{1+2} \det(A^{21}) = -\det([b]) = -b \\
(\mathrm{adj}(A))_{21} &= (-1)^{2+1} \det(A^{12}) = -\det([c]) = -c \\
(\mathrm{adj}(A))_{22} &= (-1)^{2+2} \det(A^{22}) = \det([a]) = a.
\end{aligned}$$

Thus, Cramer's formula gives the formula for the inverse of A we used earlier:

$$A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A) = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Cramer's rule: continuity of solutions and of the inverse. In general, Cramer's rule is not a time-efficient or numerically stable way to compute the solution to a system of equations. However, it is theoretically useful as we see from the following immediate corollaries of the rule:

Theorem. *Let F be $\mathbb{R}$ or $\mathbb{C}$, and let $\mathrm{GL}_n(F)$ denote the set of invertible $n \times n$ matrices over F.*

(1) The function

$$\begin{aligned}
\mathrm{GL}_n(F) &\longrightarrow F \\
A &\longmapsto A^{-1}
\end{aligned}$$

is a continuous function. In other words, the inverse of A is a continuous function of the entries of A.

(2) The solution x to the system $Ax = b$ is a continuous function of the entries of A and b.

Proof. For part (1), it suffices to show that the entries of A^{-1} are rational functions (i.e., quotients of polynomials) in the entries of A (with denominators that do not vanish for invertible A). But this follow's immediately from Cramer's rule:

$$A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A).$$

The function $A \mapsto \det(A) \in F$ is a polynomial in the entries of A (consider the permutation or Laplace expansion of the determinant), hence continuous. Hence, restricted to invertible matrices, the function $A \mapsto$

$1/\det(A)$, which gives the denominators of the entries of A^{-1}, is continuous. Similarly, the entries of $\mathrm{adj}(A)$ are polynomials in the entries of A. The result follows.

Part (2) follows since $Ax = b$ implies $x = A^{-1}b$. We've just seen that the entries of A^{-1} are quotients of polynomials in the entries of A, hence the components of x are quotients of polynomials in the entries af A on b. $\square$