MATH 201: LINEAR ALGEBRA
HOMEWORK DUE FRIDAY WEEK 7

Problem 1. Suppose that A is an invertible square matrix and P and Q are square matrices such that

$$PAQ = I.$$

Prove that

$$A^{-1} = QP.$$

Problem 2. Square matrices are called *similar* when they represent the same linear transformation, each with respect to the same starting and ending bases. In other words, $A \in F^{n\times n}$ is similar to $B \in F^{n\times n}$ if and only if there are ordered bases α and β of F^n and a linear transformation $f\colon F^n \to F^n$ such that

$$A = A^{\alpha}_{\alpha}(f) \qquad \text{and} \qquad B = A^{\beta}_{\beta}(f).$$

(a) Prove that $A, B \in F^{n\times n}$ are similar if and only if there is an invertible matrix $P \in F^{n\times n}$ such that $A = P^{-1}BP$.
(b) Prove that similarity is an equivalence relation on $F^{n\times n}$.
(c) Show that if A is similar to B, then A^k is similar to B^k for all natural numbers k.

Problem 3. Suppose that, with respect to $\alpha = \mathcal{E}_2$ and $\beta = ((1,1),(1,-2))$, the linear transformation $t\colon \mathbb{R}^2 \to \mathbb{R}^2$ is represented by the matrix

$$A^{\beta}_{\alpha}(t) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Use change-of-basis matrices to represent t with respect to the following pairs of bases:
(a) $\delta = ((0,1),(1,1))$, $\gamma = ((-1,0),(2,1))$,
(b) $\varepsilon = ((1,2),(1,0))$, $\zeta = ((1,2),(2,1))$.

Problem 4. Suppose $f\colon V \to W$ is a linear transformation. Define $f^*\colon W^* \to V^*$ by the rule

$$\phi \longmapsto f^*(\phi) = \phi \circ f.$$

(Recall that ϕ is a linear map $W \to F$, so $\phi \circ f$ makes sense as a linear map $V \to F$, *i.e.*, an element of V^*.)
(a) Prove that f^* is a linear transformation.
(b) Suppose $g\colon U \to V$ is a linear transformation. Prove that $(f\circ g)^* = g^* \circ f^*$. Also verify that $\mathrm{id}_V^* = \mathrm{id}_{V^*}$.[1]
(c) Suppose that V and W have ordered bases $\langle v_1, \dots, v_n\rangle$ and $\langle w_1, \dots, w_m\rangle$, respectively, and suppose that f has matrix $A \in \mathsf{Mat}_{m\times n}(F)$ with respect to these ordered bases. Prove that the matrix of f^* relative to $\langle w_1^*, \dots, w_m^*\rangle$ and $\langle v_1^*, \dots, v_n^*\rangle$ is the *transpose* of A, *i.e.*, $A^\top$ with

$$(A^\top)_{ij} = A_{ji}.$$

(d) Use your work from (b) and (c) to write a **short**, non-computational proof that $(AB)^\top = B^\top A^\top$.

Problem 5. The *trace* of an $n \times n$ matrix A is the sum of its diagonal elements:

$$\operatorname{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$

[1]This makes $(\ )^*$ a *constravariant functor*, but you needn't know what that means.

(a) If A and B are $n \times n$ matrices, prove that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. (You could use the definition of matrix multiplication and summation notation in your proof, or there is a slick proof that uses duals and your work from the previous problem — how does $\operatorname{tr}(A)$ compare to $\operatorname{tr}(A^\top)$?)

(b) If P is an invertible $n \times n$ matrix, prove that $\operatorname{tr}(PAP^{-1}) = \operatorname{tr}(A)$.

(c) Consider the following ordered basis for $\mathcal{M}_{2\times 2}$:

$$\alpha = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right).$$

A rote check reveals that tr is a linear transformation. Assuming this fact, compute the matrix representing the trace function $\operatorname{tr}\colon F^{2\times 2} \to F$ with respect to α for the domain and with respect to the basis $\{1\}$ for the codomain.