MATH 201: LINEAR ALGEBRA
HOMEWORK DUE FRIDAY WEEK 5

Problem 1. Let V and W be vector spaces over F, and let $f\colon V \to W$ be a linear transformation.

(a) Prove that f is injective if and only if f carries linearly independent subsets of V to linearly independent subsets of W.
(b) Suppose that f is injective and that S is a subset of V. Prove that S is linearly independent if and only if $f(S)$ is linearly independent.

Problem 2. Fix an F-vector space V. Recall from Problem 5 of the Week 5 homework that $V^* = \operatorname{Hom}(V, F)$ is the *dual* of V. We write $V^{**} := (V^*)^*$ for the *double dual* of V; its elements are linear transformations from V^* to F. The *evaluation* map $\operatorname{ev}\colon V \to V^{**}$ is the function taking $v \in V$ to $(\operatorname{ev}_v\colon V^* \to F) \in V^{**}$ where $\operatorname{ev}_v(f) = f(v)$.

(a) Prove that ev is a linear transformation.
(b) It is a fact that for any nonzero $v \in V$, there exists $f \in V^*$ such that $f(v) \neq 0$. (*Challenge*: Prove it. If V is infinite dimensional, you will need to invoke the axiom of choice.) Use this to prove that ev is injective.
(c) Use Problem 5 of Week 5 to show that $V \cong V^* \cong V^{**}$ when V is finite dimensional.[1]

Problem 3. Let

$$A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & -3 \\ 4 & 1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 & 4 \\ -1 & -2 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}.$$

Compute, if possible, the following. If it is not possible, explain why.

(a) AB,	(b) $A(2B + C)$,	(c) $A + C$,
(d) $(AB)D$,	(e) $A(BD)$,	(f) AD.

Problem 4. Let A and B be $m \times n$ matrices over F, and C an $n \times p$ matrix over F. Prove that $(A + B)C = AC + BC$. (This is called the *right distributivity* property.)

Problem 5. Let $\mathbb{R}[x]_{\leq n}$ be the vector space of polynomials in x of degree at most n with coefficients in $\mathbb{R}$. Define

$$f\colon \mathbb{R}[x]_{\leq 2} \to \mathbb{R}[x]_{\leq 3}$$

$$p(x) \mapsto \int_0^x p(t)\,dt.$$

Find the matrix representing f with respect to the bases $\{1, x, x^2\}$ for $\mathbb{R}[x]_{\leq 2}$ and $\{1, x, x^2, x^3\}$ for $\mathbb{R}[x]_{\leq 3}$.

Problem 6. Let V be a vector space over a field F. Recall that the *identity function* is the linear function $\mathrm{id}_V\colon V \to V$ by $\mathrm{id}_V(v) = v$ for all $v \in V$.

(a) Let V be a vector space of dimension n and let α be an ordered basis for V. Show that the matrix representing id_V with respect to the basis α for both the domain and the codomain is I_n (the $n \times n$ identity matrix).

[1] When V is infinite dimensional, this no longer holds. Also note that $V \cong V^*$ depends on the choice of a basis while the definition of ev does not depend on a basis; it is *canonical*.

(b) Let V and W be vector spaces of dimension n and let $f\colon V \to W$ be an isomorphism with inverse $f^{-1}\colon W \to V$. Let α and β be ordered bases for V and W, respectively. If A is the matrix representing f with respect to the bases α and β, what is the matrix for f^{-1} with respect to the bases β and α?

(c) Consider the linear transformation

$$\begin{aligned} f\colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ (x, y) &\mapsto (3x + y, -x + 4y). \end{aligned}$$

Using part (b), prove that f is an isomorphism by exhibiting its inverse using matrix calculations. Write the inverse in the form $g(x, y) = (\text{blah}, \text{blah})$.