MATH 201: LINEAR ALGEBRA
HOMEWORK DUE FRIDAY WEEK 5

Problem 1. Find the coordinates of each given vector v with respect to the ordered list of linearly independent vectors $B = \langle \beta_1, \ldots, \beta_n \rangle$. Show your work.

(a) $v = (11, -6),\ B = \langle (1,2), (-2,3) \rangle$.
(b) $v = (11, -6),\ B = \langle (1,0), (0,1) \rangle$.
(c) $v = x^2 + 7x - 5,\ B = \langle 1, (x-1), (x-1)^2 \rangle$.
(d) $v = x^2 + 7x - 5,\ B = \langle 1, x, x^2, x^3 \rangle$. (Note: $x^3 \in B$).
(e)

$$v = \begin{pmatrix} 3 & 7 \\ 8 & 11 \end{pmatrix}, \qquad B = \left\langle \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 3 \end{pmatrix} \right\rangle.$$

Problem 2. Let A be an $m \times n$ matrix with i,j-th entry A_{ij}. The *transpose* of A, denoted A^T, is the $n \times m$ matrix with i,j-th entry A_{ji}: the i-th row of A^T is the i-th column of A. Thus, for example,

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}^T = \begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix}.$$

A matrix A is *symmetric* if $A^T = A$. A matrix A is *skew-symmetric* if $A^T = -A$. A 3×3 skew-symmetric matrix has the form:

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}.$$

Let W be the set of 3×3 skew-symmetric matrices over a field F.

(a) Prove that W is a subspace of the vector space of all 3×3 matrices over F.
(b) Give a basis for W.
(c) What is $\dim(W)$?

Problem 3. Define the following matrix over the real numbers:

$$M = \begin{pmatrix} -14 & 56 & 40 & 92 \\ 8 & -32 & -23 & -53 \\ 6 & -24 & -17 & -39 \\ -1 & 4 & 3 & 7 \end{pmatrix}.$$

(a) What is the reduced echelon form for M? (You do not need to show your work for this. Thinking a bit about your choices will save work.)
(b) Compute (i) a basis for the row space of M and (ii) a basis for the column space of M using the algorithm presented in class on Monday of Week 4. (Make sure you follow the algorithm precisely. The solution is then unique.)

Problem 4. (a) Prove that there exists a linear transformation $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ such that $f(1,1) = (1,0,2)$ and $f(2,3) = (1,-1,4)$. What is $f(8,11)$?
(b) Is there a linear transformation $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ such that $f(1,2,1) = (2,3)$, $f(3,1,4) = (6,2)$ and $f(7,-1,10) = (10,1)$? Explain your reasoning.

Problem 5. Recall that $\mathrm{Hom}(V, W)$ denotes the vector space of linear transformations $V \to W$ under pointwise addition and scalar multiplication.[1] When $W = F = F^1$, the vector space $V^* := \mathrm{Hom}(V, F)$ is the called the *dual* of V. Suppose that V has ordered basis $\langle v_1, \ldots, v_n \rangle$. Prove that V^* has basis $\langle v_1^*, \ldots, v_n^* \rangle$ where v_i^* satisfies

$$v_i^*(v_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

(Your solution should include a justification of why v_i^* is a well-defined element of V^*.)

Problem 6. For the following functions f:

(i) prove that f is a linear transformation,
(ii) find bases for $\ker(f)$ and $\mathrm{im}(f)$, and
(iii) compute the nullity and the rank.

(a) $f \colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by $f(x, y, z) = (x - y, 2z)$.
(b) $f \colon \mathbb{R}^2 \to \mathbb{R}^3$ defined by $f(x, y) = (x + y, 0, 2x - y)$.
(c) $f \colon \mathbb{R}[x]_{\leq 2} \to \mathbb{R}[x]_{\leq 3}$ defined by $f(p(x)) = x \cdot p(x) + p'(x)$. Here, $p'(x)$ denotes the usual derivative from one-variable calculus.

(Recall that $F[x]_{\leq n}$ denotes the vector space of polynomials with coefficients in F of degree less than or equal to n. One basis for it is $\{1, x, x^2, \ldots, x^n\}$, and hence, it has dimension $n + 1$.)

[1] *I.e.*, $(f + g)(v) = f(v) + g(v)$ and $(\lambda f)(v) = \lambda f(v)$ for all $f, g \in \mathrm{Hom}(V, W)$ and $\lambda \in F$.