MATH 201: LINEAR ALGEBRA
HOMEWORK DUE FRIDAY WEEK 3

Problem 1. Prove that the the following sets and operations do not form vectors spaces. As usual, to disprove something, you need to provide a concrete counterexample, ideally as simple as possible.

(a) $V = \mathbb{R}^2$, with

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 y_2 \end{pmatrix} \quad \text{and} \quad r \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} r x_1 \\ y_1 \end{pmatrix}.$$

(b) $V = \mathbb{R}^2$, with

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} \quad \text{and} \quad r \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} r x_1 \\ 0 \end{pmatrix}.$$

(c) $V = \{(x, y) \in \mathbb{R}^2 : x + 2y = 3\}$ with the usual addition and scalar multiplication for vectors in $\mathbb{R}^2$.

Problem 2. Here are two templates for showing a subset W of a vector space V over a field F is a subspace:

Proof 1. First note that $0 \in W$ since ___________. Hence, $W \neq \emptyset$. Next, suppose that $u, v \in W$. Then ___________. Hence, $u+v \in W$. Now suppose $\lambda \in F$ and $w \in W$. Then ___________. Therefore, $\lambda w \in W$. □

Proof 2. First note that $0 \in W$ since ___________. Hence, $W \neq \emptyset$. Next, suppose that $\lambda \in F$ and $u, v \in W$. Then ___________. Hence, $\lambda u + v \in W$. □

Use one of these two templates for each of the following exercises.

(a) Show that $W = \{(x, y, z) \in \mathbb{R}^3 : 3x + 2y - z = 0\}$ is a subspace of $\mathbb{R}^3$.

(b) Show that the set $W = \{f : \mathbb{R} \to \mathbb{R} : f(t) = f(-t) \text{ for all } t \in \mathbb{R}\}$ is a subspace of the vector space of real-valued functions of one variable. (Hint: you will need to carefully use the definitions given in Example 1.12, p. 90, of the text.)

Problem 3. In each of the following:

- Determine whether the given vector v is in the span of the set S by creating a relevant system of linear equations in the usual form

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

and then row reducing the corresponding augmented matrix for the system.

- If v is in the span of S, then explicitly write v as a linear combination of the vectors in S.

Assume we are working over the field $\mathbb{Q}$ of rational numbers.

(a) $v = (0, -1, -6)$, $S = \{(1, 0, -1), (2, 1, 3), (4, 2, 5)\}$.

(b) $v = (1, 2, 4)$, $S = \{(1, 4, 7), (2, 5, 8), (3, 6, 9)\}$.

(c) $v = x^3 - 13x^2 + 7x + 27$, $S = \{x^3 + 3x^2 - 2, x^3 + x^2 + 4x + 1, 2x^2 + x + 4\}$.

(d) $v = \begin{pmatrix} 9 & 12 \\ 10 & 9 \end{pmatrix}$, $S = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix} \right\}$.

Problem 4. Determine whether the following sets are linearly dependent or linearly independent.

(a) $\{x^3 + 2x^2, -x^2 + 3x + 1, x^3 - x^2 + 2x - 1\}$ in $\mathbb{R}[x]$.
(b) $\{(1, -1, 2), (1, -2, 1), (1, 1, 4)\}$ in $\mathbb{R}^3$.
(c) $\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ in $\mathbb{R}^3$.
(d) $\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ in $(\mathbb{Z}/2\mathbb{Z})^3$ (where $\mathbb{Z}/2\mathbb{Z}$ is the field with two elements).

Problem 5. Let F be a finite field with $|F| = q$.[1] Let $S = \{u_1, \dots, u_n\}$ be a set of linearly independent vectors in an F-vector space. Determine the cardinality of $\operatorname{span} S$.

Problem 6. Fix a field F. Using only the material we have developed in class thus far, show that a set

$$S = \{(a, b), (c, d)\} \subseteq F^2$$

is linearly independent if and only if

$$ad - bc \neq 0.$$

[1]Here $|F|$ denotes the *cardinality* of F, *i.e.*, the number of elements in the set F. In abstract algebra, you will discover that q must be of the form p^n for some prime p.