CHAPTER 3

Writing mathematics I

We have a habit in writing articles published in scientific journals to make the work as finished as possible, to cover up all the tracks, to not worry about the blind alleys or describe how you had the wrong idea first, and so on.

Richard Feynman, Nobel Lecture, 1966

As a lecturer my toughest initial task in turning enthusiastic students into able mathematicians is to force them (yes, force them) to write mathematics correctly. Their first submitted assessments tend to be incomprehensible collections of symbols, with no sentences or punctuation. 'What's the point of writing sentences?', they ask, 'I've got the correct answer. There it is – see, underlined – at the bottom of the page.' I can sympathize but in mathematics we have to get to the right answer in a rigorous way and we have to be able to show to others that our method is rigorous.

A common response when I indicate a nonsensical statement in a student's work is 'But you are a lecturer, you know what I meant.' I have sympathy with this view too, but there are two problems with it.

(i) If the reader has to use their intelligence to work out what was intended, then the student is getting marks because of the reader's intelligence, not their own intelligence.[1]

(ii) This second point is perhaps more important for students. Sorting through a jumble of symbols and half-baked poorly expressed ideas is likely to frustrate and annoy any assessor – not a good recipe for obtaining good marks.

My students performed well at school and are frustrated at losing marks over what seems to them unimportant details. However, by the end of the year they generally accept that writing well has improved their performance. You have to trust me that this works! Besides, writing well in any subject is a useful skill to possess.

[1] To be honest, students don't mind this!

Writing well is good for you

Writing well

There are many reasons for writing – you might be making notes for future use or wish to communicate an idea to another person. Whatever the reason, writing mathematics is a difficult art and requires practice to produce clear and effective work.

Good writing is clearly important if you wish to be understood, but it has a bonus: it clarifies for you the material being communicated and thus adds to your understanding. In fact, I believe that if I can't explain an idea in writing then I don't understand it. This is one reason why writing well helps you to think like a mathematician.

Generally, we write to explain to another person, so have this person in mind. Two points to remember:

- Have mercy on the reader. Do not make it difficult for them – particularly someone marking your work.
- The responsibility of communication lies with you. If someone at your level can't understand it, then the problem is with your writing!

What follows is a collection of ideas on how to improve your writing. The ideas presented have been tried and tested with students over many years and are not merely theoretical ideas. They may seem troublesome and pedantic, but if you follow them you will produce clearer explanations, and hence gain more marks in assessments.

It should be noted that there is a huge difference between finding the answer to a problem and presenting it. These rules apply to the final polished product. When trying to solve a problem or do an exercise it is acceptable to break all these rules. What is important is that they are followed when writing up the solution for someone else to read.

An example

In a geometry course I stated the Cosine Rule.

> **Cosine Rule:** Suppose that a triangle has edges of length a, b and c with the angle opposite a equal to θ. Then,
>
> $$a^2 = b^2 + c^2 - 2bc\cos\theta.$$

If you have not met this before, then this is a good chance to 'Check the text' as described on page 18. Try drawing some pictures and trying some examples. More techniques for investigating such a statement will be found in Chapter 16.

The cosine rule is a useful result which can be regarded as a generalization of Pythagoras' Theorem when we take $\theta = \pi/2$. (Check the text!) During the geometry course I proved this formula in the case that θ was an acute angle and left the case of an obtuse angle as an exercise. Figure 3.1 shows one solution I received. We will refer to this as we proceed. As an exercise take a look at it and try to spot as many errors as possible. Does it make sense? Is it easy to read? Most importantly, is it right?

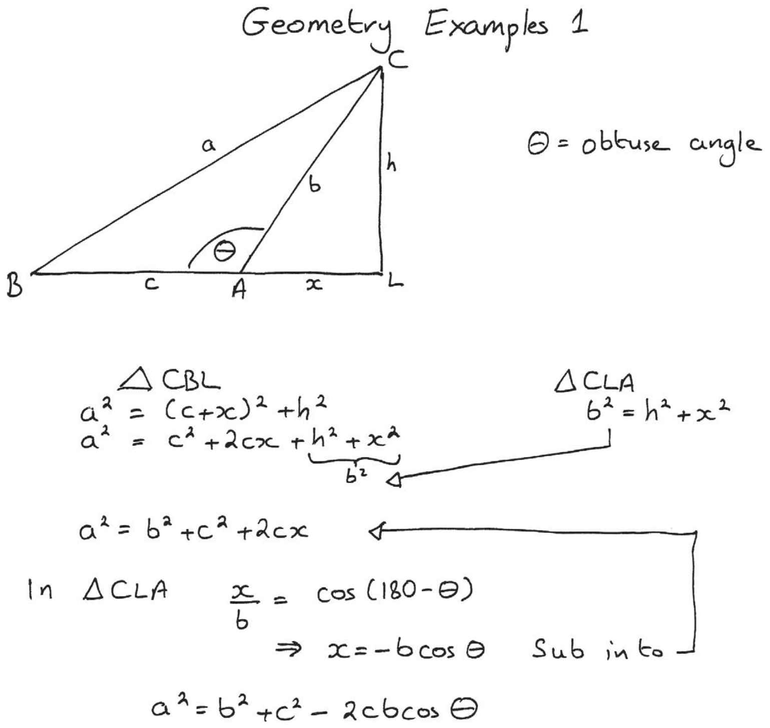


Figure 3.1 Student's proof of the Cosine Rule

Basic rules

The primary rule is that you should write in simple, correctly punctuated sentences. Let's put some more detail on this.

Write in sentences

Write in sentences. Write in sentences. And once more to really hammer it home: Write in sentences.

This advice has precedence over all others and is the one that can really change the way you present your work.

One of most common erroneous beliefs of the novice mathematician is that because mathematics is a highly symbolic language we need only provide a list of symbols to answer a problem. This is wrong, symbols are merely shorthand for certain concepts; they need to be incorporated into sentences for there to be any meaning.

Consider this student's answer to an exercise on finding the solution of a set of equations:

> '$0 = 1$, $\therefore$ no solutions, empty set ($\emptyset$).'

It is obvious what the student meant: 'Since the equations reduce to the equation '$0 = 1$', which doesn't have any solutions, the solution set is empty.' This vital fact – that no solutions exist – is certainly included. The student also showed that he knows that the empty set is denoted by ∅. However, the inclusion of this symbol is unnecessary; it serves no purpose.

But what he wrote is not a sentence – it is a string of symbols and conveys no meaning in itself.

The answer could be better expressed as

> 'Since the equation $0 = 1$ is present, the system of equations is inconsistent and so no solutions exist.'

We could add 'That is, the solution set is empty', but it is not necessary. Understanding is clearly shown in this answer, and so more marks will be forthcoming.

All the other usual rules of written English apply, for example the use of paragraphs and punctuation. The rules of grammar are just as important: every sentence should have a verb, subjects should agree with verbs, and so on.

Let us look at the example in Figure 3.1 of the proof of the cosine formula. Examine the first two lines below the student's diagram.

$\triangle$ CBL $a^2 = (c+x)^2 + h^2$ $\triangle$ CLA $b^2 = h^2 + x^2$

If I read from left to right in the standard fashion, I read

$$\Delta CBL\ \Delta CLA\ a^2 = (c+x)^2 + h^2\ b^2 = h^2 + x^2.$$

Now what does that mean? It is obvious what is intended. But why should we have to work out what was intended? It would be better to say what was meant from the start:

> In triangle ΔCBL we have $a^2 = (c+x)^2 + h^2$ and in ΔCLA we have $b^2 = h^2 + x^2$.

This is now a proper sentence. As an aside, notice how I explained my notation Δ by using the word 'triangle'.

Now look at the words after the $\implies$ sign:

$\frac{x}{b} = \cos(180 - \theta)$

$\Rightarrow x = -b\cos\theta$ Sub into

This is a perfect example of where we can understand what the student had intended but it is not well written. It is much clearer as

> $\ldots\ x = -b\cos\theta$. Substituting this into …

Use punctuation

The purpose of punctuation is to make the sentence clear. Punctuation should be used in accordance with standard practice. In particular, all sentences begin with a capital letter and end with a full stop. The latter holds even if the sentence ends in a mathematical expression. For example,

> 'Let $x = y^4 + 2y^2$ Then x is positive.'

needs a full stop after the expression $y^4 + 2y^2$ as it is obvious that the second part is a new sentence – it begins with a capital letter. This is true for a list of equal expressions:

$$\begin{aligned} x &= y^2 + 2y \\ &= y(y+2) \end{aligned}$$

This should end with a full stop. Note that some authors do not adhere to this rule of punctuation. They are wrong.[2]

Mathematical expressions need to be punctuated. For example,

> 'Let $x = 4a + 3b$ where $a \in \mathbb{R}$ $b \in \mathbb{Z}$'

should have commas like so

> 'Let $x = 4a + 3b$, where $a \in \mathbb{R}$, $b \in \mathbb{Z}$.'

Notice the three commas and the final full stop in the following example.

$$\text{Let } f(x) = \begin{cases} x^2, & \text{for } x \geq 0, \\ 0, & \text{for } x < 0. \end{cases}$$

Look at the example of the proof of the cosine formula. As you can see there is is no punctuation! Presumably a sentence starts at 'In ΔCLA ...' but it is not proceeded by a full stop so who knows?

Keep it simple

Mathematics is written in a very economical way. To achieve this, use short words and sentences. Short sentences are easy to read. To eliminate ambiguities avoid complicated sentences with lots of negations.

Consider the following hard-to-read example:

> 'The functions f and g are defined to be equal to the function defined on the set of non-positive integers given by x maps to its square and x maps to the negative of its square respectively.'

This would be better as:

> 'Let $\mathbb{Z}^{\leq 0} = \{\dots, -5, -4, -3, -2, -1, 0\}$ be the set of non-positive integers. Let $f : \mathbb{Z}^{\leq 0} \to \mathbb{R}$ be given by $f(x) = x^2$ and $g : \mathbb{Z}^{\leq 0} \to \mathbb{R}$ be given by $g(x) = -x^2$.'

Note that we separated the definition of the domains of the maps into a separate sentence.

[2] A number of people think this is a controversial statement. 'What does it matter, as long as you are consistent?' Well, we could apply that argument to any sentence and we can get rid of all full stops! The majority opinion is that sentences end with a full stop – go with that.

Also we defined the set in words and clarified by writing it in a different way. The definitions of f and g are mixed together in the first sentence due to the use of ‘respectively’, while in the second sentence they are separated and defined using symbols. Sometimes using symbols is clearer, sometimes not; see page 28.

Expressing yourself clearly

The purpose of writing is communication – you are supposed to be transferring a thought to someone else (or yourself at a later date). Unfortunately – and I have lots of experience of this – it is easy to communicate an incorrect or unintended idea. The following advice is offered to prevent this from happening.

Explain what you are doing – keeping the reader informed

Readers are not psychic. It is crucial to explain what you are doing. To do this imagine that you are giving a running commentary. As stated earlier, it is not sufficient to produce a list of symbols, formulas, or unconnected statements. A good explanation will help gain marks as it demonstrates understanding.

You can introduce an argument by saying what you are about to do, e.g.

> ‘We now show that X is a finite set’,
> ‘We shall prove that . . .’.

Similarly you can end by

> ‘This concludes the proof that X is a finite set’, or
> ‘We have proved . . .’.

Make clear, bold assertions. Avoid phrases like ‘it should be possible’; either it is possible or it isn’t, so claim ‘it is possible’. Be positive.

Of course, avoid going to the extreme of explaining every last detail. A balance, which will come from practice and having your written work criticized, needs to be struck.

If we look at the end of the example in Figure 3.1, then we see the following.

$\Rightarrow x = -b\cos\theta$ Sub into

$a^2 = b^2 + c^2 - 2cb\cos\theta$

This ending would be better as

> ‘. . . $x = -b\cos\theta$. Substituting this into the above we deduce that $a^2 = b^2 + c^2 - 2cb\cos\theta$.’

This is certainly much better as it implicitly makes the claim that what we had to prove has been proved. Otherwise it may look like we wrote the cosine formula at the end to fool the marker into thinking that the solution had been given. Also, using the word ‘deduce’ in the final sentence explains where the result came from.

Explain your assertions

Rather than merely make an assertion, say where it comes from. That is, use sentences containing

'as, because, since, due to, in view of, from, using, we have,' and so on.

For example,

'Using Theorem 4(i), we see that the solution set is non-empty'

is obviously preferable to

'The solution set is non-empty,'

and

'$x^3 > 0$ because x is positive'

is better than the bare

'$x^3 > 0$'

since, for a general $x \in \mathbb{R}$, we don't have $x^3 > 0$. The point is that the reader may be misled into thinking the statement is 'obviously false' if they had forgotten that x was positive. It doesn't hurt to include such helpful comments.

Another example is to say when a rule has been used:

'$f'(x) = 2x \cos(x^2)$ by the Chain Rule.'

In this way, you demonstrate your understanding.

Returning to the first few lines of the student's proof of the cosine formula in Figure 3.1

$\triangle CBL$ $\quad\quad\quad\quad$ $\triangle CLA$

$a^2 = (c+x)^2 + h^2$ $\quad\quad\quad\quad$ $b^2 = h^2 + x^2$

we have already seen that it would be better to have said

'In triangle ΔCBL we have $a^2 = (c+x)^2 + h^2$ and in ΔCLA we have $b^2 = h^2 + x^2$.'

But what about the next line? It says simply

$a^2 = c^2 + 2cx + h^2 + x^2$

Is this a deduction from the diagram? Certainly the first two equalities were, i.e. $a^2 = (c+x)^2 + h^2$ and $b^2 = h^2 + x^2$. In this case the line is not deduced from the diagram but from the first equation by expanding the bracket. So we should say so.

Expanding the brackets we get $a^2 = c^2 + 2cx + h^2 + x^2$.

We'll see that it is not necessary to phrase it this way when we look at the next line:

$a^2 = b^2 + c^2 + 2cx$

This comes from substituting the second equation, $b^2 = h^2+x^2$, into the expanded version of the first, $a^2 = c^2 + 2cx + h^2 + x^2$. Let's say so.

> 'In triangle ΔCBL we have $a^2 = (c + x)^2 + h^2$ and in ΔCLA we have $b^2 = h^2 + x^2$. Expanding this first equation and substituting in b^2 from the second we get $a^2 = b^2 + c^2 + 2cx$.'

Note that we have left out the expansion of the brackets. You can include it if you wish but the calculation is so trivial that it is not worth the ink. The reader can check it themselves if they don't believe us.

Say what you mean

In any writing, saying what you mean is important – and difficult. Precise use of grammar can help in this task.

The first rule is that the reader should not have to deduce what you mean from context; all the necessary information should be there. Nothing should be ambiguous.

The true mathematician is pedantic, and requires that mathematics is precise. Without precision mathematics is nothing. Without it we cannot build with one concept placed on top of another. If one of the ideas is vague or open to different interpretations by different parties, then errors can creep in and the endeavour is unsound. So, be precise!

As an example, use the quantifiers 'some' and 'all'. Rather than say

> '$f(x) = 5$',

which is ambiguous – the reader may ask 'Is it for one x? At least one x? All x?' – say

> '$f(x) = 5$ for some $x \in \mathbb{R}$', or '$f(x) = 5$ for all $x \in \mathbb{R}$',

depending on the situation.

More will be said in Chapter 10 on quantifiers to explain the importance of precision in this area.

Using symbols

We now come to tips concerning symbols. There is no escaping that mathematics is highly symbolic, but using lots of mathematical symbols does not make an argument a mathematical one.

Words or symbols?

Symbols are shorthand. For example, a famous theorem by Euler in the theory of complex numbers,

$$e^{2\pi\sqrt{-1}} = 1,$$

is concisely expressed in symbols.[3] The equivalent statement written out in words is less impressive:

> 'The exponential of two times the circumference of a circle divided by its diameter times the square root of minus one is equal to one.'

However, a good general rule of thumb is to use words. For example, use 'therefore' rather than the $\therefore$ symbol. Very few books use it. Similarly,

> 'x is a rational number $\Rightarrow x^2$ is real',

can be written as

> 'x is a rational number implies that x^2 is real.'

In some sentences it is best to avoid mixing symbols and words. For example,

> 'The answer $= 1$'

should be written as

> 'The answer equals 1.'

Otherwise we produce sentences like

> 'The number of people aged over 40 = 5',

which reads all right, but the eye is drawn to the (erroneous) expression $40 = 5$.

Small numbers used as adjectives should be spelled out, for example,

> 'the two sets'.

They should be in numerals when used as names or numbers, as in

> 'Lemma 3' and '... has mean equal to 23'.

Another example:

> 'One of the roots is 3.'

An exception is the number 1, which traditionally can be either.

Note that symbols which are similar can cause confusion: clearly differentiate between $\in$ and ε. The former usually denotes membership of a set and the latter is the Greek letter epsilon, but be aware that other writers use them the other way round.

As noted earlier we rewrote the first few lines and to include the standard notation ΔCBL:

> 'In triangle ΔCBL we have $a^2 = (c+x)^2+h^2$ and in ΔCLA we have $b^2 = h^2+x^2$.'

[3] This is a great theorem – it relates many great numbers, e, π, the square root of -1 and of course two important natural numbers: 1 and the only even prime, 2. In a poll of mathematicians (*Mathematical Intelligencer*, Vol. 12 no. 3, 1990, pp. 37–41), this theorem was voted the most beautiful theorem in mathematics.

Equals means equals

The equals sign, =, is one of the most common in mathematics, and one of the earliest learned by children. Despite this, or maybe because of it, it is still badly abused.

Let's go back to the beginning and note that, in using the equals sign, we are asserting that the two objects on either side are *exactly the same* – being almost the same is not enough, being close is not enough, being similiar in a poor light and from a distance is not enough! For numbers this idea of equality should be second nature. But it holds for other objects. Thus remember:

Equals means equals.

One consequence is that if on one side of the sign there is a function, then on the other side there must be a function. If on one side there is a set, then on the other there must be a set.

In answer to a question on factorising numbers into primes, one of my students wrote:

Factors: $6 = 2$ and 3.

Leaving aside the observations that this is a poor sentence and '$6 = 2$' is not good on the eye, the idea expressed is false. True, 6 is *equal to the product* of 2 and 3, and so has 2 and 3 as factors, but it is not *equal* to '2 and 3'. A better answer is:

The prime factors of 6 are 2 and 3.

Similarly, consider the exercise, 'Find the derivative of x^3', the answer is not

$$x^3 = 3x^2.$$

Now, this does give a mathematical expression, in fact an equation, but it is not what the student wanted to assert. One correct way to write it is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\right) = 3x^2.$$

A very common mistake is to use the equals as a link from one line to the next, almost like a sign saying this is the next part of the process. The correct way of displaying results is given next.

Displaying results with the equals sign

If an expression is short, we show working by writing across the page. For example, $(x + 3)^2 = (x + 3)(x + 3) = x^2 + 6x + 9$.

For a longer calculation it is traditional to write down the page like so:[4]

$$\begin{aligned}(x+3)^2 + x^2 &= (x+3)(x+3) + x^2\\ &= x^2 + 6x + 9 + x^2\\ &= 2x^2 + 6x + 9.\end{aligned}$$

[4] Unfortunately, this violates our rule on punctuation, but we do it anyway as it is practical and traditional.

Sometimes we need to indicate where a particular result came from. Avoid interrupting the flow of the argument like so:

$$\begin{aligned} &= x^2 + 5y \\ \text{by theorem 6 } &= x^2 + 25 \ldots \end{aligned}$$

If the details of why a particular step is true need to be included, then do the following. For the sake of argument suppose that $y = 3$ by Theorem 4.6. Then we write

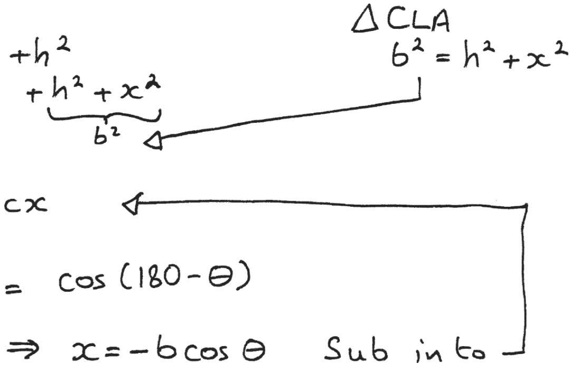
$$\begin{aligned} x^2 + 4x + y &= x^2 + 4x + 3, \text{ by Theorem 4.6,} \\ &= (x+1)(x+3) \ldots \end{aligned}$$

Note the punctuation after the symbolic expression on the first line and after the mention of the theorem. It doesn't read well, but is clear on the page.

Don't draw arrows everywhere

If a result requires an earlier one, it is tempting to draw a long arrow to point to it. Don't do this on aesthetic grounds. Instead, give the required result a name, number or symbol, so you can refer to it.

Our example in Figure 3.1 uses arrows.



We can change this to

> 'Expanding this first equation and substituting in b^2 from the second we get $a^2 = b^2 + c^2 + 2cx$. (∗)
>
> $\vdots$
>
> $\ldots \implies x = -b\cos\theta$. Substituting this into (∗) we deduce that $a^2 = b^2 + c^2 - 2cb\cos\theta$.'

Exercise 3.1

Rewrite the proof of the the Cosine Rule so that it follows the suggestions given.

Finishing off

Proofread

Always proofread your work. That is, read through it looking for errors. These could be typographical errors (also known as typos), where the wrong character is used, e.g. cay instead of cat, or spelling mistakes, e.g. parrallel instead of parallel, grammatical mistakes, e.g. ‘A herd of cows are in the field’, or even mathematical errors.

Read your work slowly. Reading aloud can help catch many errors as it stops you skimming. Get someone else to read your work as you will often read what you think is there, rather than what actually is there. If your checker misses mistakes, then you are not allowed to blame them. The final responsibility always rests with the writer!

A useful proofreading method is to concentrate on one aspect of proofreading at a time. That is, read through first for accuracy, i.e. is it true? Next, check for spelling, typos, are all the brackets closed?, etc. After that check that the order of the material is correct and that it flows as you read it.

Reflection

Reflection is an important part of the writing process. Put your work away for some time and come back to it with a fresh eye. Obviously, this is not possible for work with tight deadlines, but can be done with project work.

When reading through again, ask ‘What can I take away?’ (aim for economy of words) and ‘What can I add?’ (more examples might clarify). For the former remove unnecessary words and sentences. Also ask: ‘Are all the symbols explained and are they necessary? Does it say what I mean and is it simple? Is it more than just a collection of symbols?’ And of course, most importantly, ‘Did I write in sentences?’

Exercises

Exercises 3.2

(i) **The Sine Rule:** Suppose that we have a triangle with sides of length a, b and c with the angles opposite these sides labelled α, β and γ respectively. Then

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}.$$

In an exam a student answered the question ‘State and prove the Sine Rule’ with the following:

Figure 3.2 Student's proof of the Sine Rule

Rewrite this answer so that it is correctly written and easily comprehended.

(ii) If you know how to find maxima and minima as well as curve sketching you should rewrite the following answer to the exercise 'Find the maximum and minimum values of the function $f(x) = 2x^3 - 12x^2 + 18x$ and sketch its graph.'

$$f = 2x^3 - 12x^2 + 18x$$

$$= 6x^2 - 24x + 18 \Rightarrow \quad x = \frac{24 \pm \sqrt{24^2 - 4 \times 18 \times 6}}{2 \times 6}$$

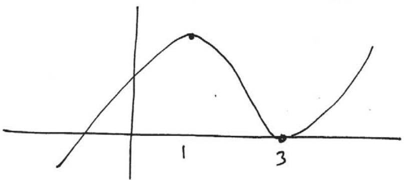
$$\frac{24 \pm \sqrt{144}}{12}$$

$$2 \pm 1$$

$$1, 3.$$

$$\frac{d^2y}{dx^2} = 12x - 24 \Rightarrow \quad \frac{d^2y}{dx^2} = 12 \times 1 - 24 = -12 < 0 \quad \text{max}$$

$$\frac{d^2y}{dx^2} = 12 \times 3 - 24 = 12 > 0 \quad \text{min}$$



$$y = 2 - 12 + 18 = 8$$

$$y = 2 \times 27 - 12 \times 9 + 18 \times 3$$

$$= 0.$$

Figure 3.3 Student's answer to finding maximum and minimum values of $f(x) = 2x^3 - 12x^2 + 18x$

(iii) Find some of your old mathematics exercises and rewrite them so that the exposition is crystal clear. You can also take examples from friends.

Summary

- Write in simple, punctuated sentences.
- Keep it simple.
- Explain what you are doing.
- Explain your assertions.
- Say what you mean.
- In general, use words rather than symbols.
- Use equals properly – equals means equals.
- Don't draw arrows everywhere – use symbols or numbers to identify equations.
- Proofread.
- Reflect.