MATH 113: DISCRETE STRUCTURES
PRACTICE FINAL EXAM

Indicate whether each of the following statements is true or false by legibly writing *True* or *False* in the space provided. (Do not just write *T* or *F* as these letters are sometimes hard to distinguish.) No work or justification required.

1. At any given moment in time, six consecutive working stoplights (with red, yellow, and green lights only) can be in one of 729 configurations. ________

2. Let S be a sample space with probability distribution P. Two events A, $B \subseteq S$ are mutually exclusive, i.e., $A \cap B = \emptyset$, if and only if $P(A)P(B) = P(A \cap B)$. ________

3. A set with $1,000,000,000$ elements has the same number of odd-sized and even-sized subsets. ________

4. There is an integer x such that $3x \equiv 1 \pmod{42}$. ________

5. The integer $370000001^{36} - 1$ is a multiple of 37. ________

6. If X and Y are random variables on the same probability space S, then $E(2X - \frac{1}{3}Y) = 2E(X) - \frac{1}{3}E(Y)$. ________

7. For any integer $n > 0$, the complete graph on $2n + 1$ vertices, K_{2n+1}, has an Eulerian circuit (i.e., a closed Eulerian walk). ________

8. The number of full binary trees with $n + 1$ leaves is equal to the number of words of length n in the alphabet $\{E, N\}$ for which, reading left-to-right, the number of N's never exceeds the number E's. ________

9. There are 125 labeled trees with 5 vertices. ________

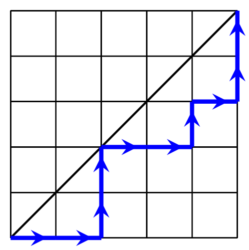
10. Every tree with $1,000,001$ vertices has $1,000,000$ edges. ________

Question 1. (a) Use the Euclidean algorithm to compute the $\gcd$ of 31 and 131.
(b) Find integers s and t such that $\gcd(31, 131) = 31s + 131t$.
(c) Does 31 have a multiplicative inverse in $\mathbb{Z}/131\mathbb{Z}$? Either prove that it does not, or compute its inverse.

Question 2. Let S be a sample space with probability distribution P. Suppose that $A, B \subseteq S$ are events such that $P(A), P(B) \neq 0$ and $P(A|B) = P(A)$. Prove that $P(B|A) = P(B)$.

Question 3. If a player rolls fifteen fair six-sided dice (with sides labeled $1, 2, \ldots, 6$), what is the expected number of 6's that will appear? Explain your work: what principle are you using?

Question 4. Consider the Dyck path



Using the bijections from class, find the corresponding full binary tree.

Question 5. (a) Use Fermat's little theorem to find the remainder of 3^{100} when divided by 7.
(b) Compute the value $\varphi(20)$ of Euler's totient function at 20.
(c) Use Euler's theorem to find the remainder of 9^{1000} when divided by 20.

Question 6. Find all integers x satisfying the following system of congruences:

$$2x \equiv 5 \pmod 7$$
$$3x \equiv 4 \pmod 8.$$

Hint: Multiplicative inverses + Sunzi's theorem.

Question 7. Consider the probability space $\mathfrak{S}_{2n}$ of permutations of $[2n] = \{1, 2, \ldots, 2n\}$ with the uniform probability distribution. For $\sigma \in \mathfrak{S}_{2n}$, let $X(\sigma)$ be the number of indices i such that $\sigma(i) > 2i$, that is,

$$X(\sigma) = |\{i \in [2n] \mid \sigma(i) > 2i\}|.$$

(a) Define the indicator random variables: X_j, $j = 1, \ldots, 2n$, where $X_j(\sigma) = 1$ if $\sigma(j) > 2j$ and equals 0 otherwise. What is the expected value $E(X_j)$ for $j = 1, \ldots, 2n$?
(b) What is the expected value $E(X)$?