**MATH 113: DISCRETE STRUCTURES**
**PRACTICE EXAM 3**

*Question* 1. Recall that a *full binary tree* is a rooted tree in which every non-leaf vertex has exactly two children (a left child and a right child). Let $f_n$ denote the number of full binary trees with $n+1$ leaves, so that $f_0 = 1$, $f_1 = 1$, $f_2 = 2$, $f_3 = 5$.

(a) Prove that $f_n$ satisfies the Catalan recurrence:

$$f_0 = 1 \qquad \text{and} \qquad f_{n+1} = \sum_{k=0}^{n} f_k\, f_{n-k} \quad \text{for all } n \geq 0.$$

(*Hint*: Consider the left and right subtrees of the root.)

(b) Conclude that $f_n = C_n$ for all $n \geq 0$, where $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$ is the $n$-th Catalan number.

*Question* 2. Two fair six-sided dice (faces numbered 1 through 6) are rolled. Let $A$ be the event that the *sum* of the two values equals 7, and let $B$ be the event that *at least one* of the two dice shows a 5 or a 6.

(a) Compute $P(A)$.
(b) Compute $P(B)$.
(c) Compute $P(A \cap B)$.
(d) Are $A$ and $B$ independent? Justify your answer.
(e) Compute $P(A \mid B)$ and give a brief intuitive explanation of why the answer is larger or smaller than $P(A)$.

*Question* 3. Suppose we flip a fair coin $n$ times, where $n \geq 2$.

(a) For $1 \leq j \leq n$, let $I_j$ be the indicator random variable for the event that flip $j$ is heads. Write the total number of heads $X$ as a sum of indicator random variables, and use linearity of expectation (Theorem 182) to compute $E(X)$. Justify each step.
(b) Let $Y$ be the number of indices $j \in \{1, 2, \ldots, n-1\}$ such that flip $j$ *and* flip $j+1$ are *both* heads. Use indicator random variables and linearity of expectation to compute $E(Y)$.
(c) Now suppose instead that the coin is *biased*: it lands heads with probability $p$ and tails with probability $1-p$, where $0 < p < 1$. The flips are still independent. Using the same approach as in (b), find the expected number of *HT* patterns (a heads on flip $j$ immediately followed by a tails on flip $j+1$) in $n$ flips of this biased coin.

*Question* 4. You *do not* need to provide justifications for your answers below.

(a) Compute the Catalan numbers $C_0, C_1, C_2, C_3, C_4$.
(b) How many full binary trees have exactly 5 leaves?
(c) How many balanced parenthesizations of length 8 are there?
(d) List the 5 complete parenthesizations of the four-factor product $abcd$.
(e) What is the probability of getting exactly 3 heads in 8 fair coin flips?
(f) If $X$ is a binomial random variable with parameters $n = 12$ and $p = 1/3$, what is $E(X)$?
(g) If $X$ is a geometric random variable with success probability $p = 1/5$, what is $E(X)$?
(h) Events $A$ and $B$ satisfy $P(A) = 1/3$, $P(B) = 1/2$, and $P(A \cup B) = 2/3$. What is $P(A \cap B)$? Are $A$ and $B$ independent?

(i) State the Law of Total Probability (Theorem 175) for two mutually exclusive events $A$ and $A^c$ whose union is the entire sample space.

**BONUS**

You should only attempt the bonus if you have completed the rest of the practice exam.

*Question* 5 (Bonus). For a positive integer $n$, let $\mathfrak{S}_n$ be the set of permutations of $[n] = \{1, 2, \ldots, n\}$. We say that $\pi(i)$ is a *left-to-right maximum* of $\pi \in \mathfrak{S}_n$ if $\pi(i) > \pi(j)$ for all $j < i$. (By convention, $\pi(1)$ is always a left-to-right maximum.) For instance, the permutation $\pi : 1 \mapsto 3,\ 2 \mapsto 1,\ 3 \mapsto 4,\ 4 \mapsto 2,\ 5 \mapsto 5$ has left-to-right maxima $3, 4, 5$ (at positions $1, 3, 5$).

If we select a permutation uniformly at random from $\mathfrak{S}_n$, what is the expected number of left-to-right maxima?

*Hint*: For each position $i$, find the probability that $\pi(i)$ is a left-to-right maximum. What is the probability that $\pi(i)$ is the largest value among $\pi(1), \ldots, \pi(i)$?