PROBLEM 1. A group of 17 people stack their books in 11 piles of equal size, each containing more than one book, and an additional pile containing 6 books. They collect the books and this time stack them into 17 equally-sized piles, with none left over. What is the smallest number of books they could have had? [Hint: -3 is the multiplicative inverse of 11 modulo 17.]

SOLUTION: We are solving for x satisfying

$$x \equiv 6 \pmod{11}$$
$$x \equiv 0 \pmod{17}$$

Thus, we need to find the smallest $k > 1$ such that $6 + 11k \equiv 0 \pmod{17}$, or

$$11k \equiv -6 \pmod{17}.$$

Since $-3 \cdot 11 \equiv 1 \pmod{17}$, we multiply through by -3 to solve for k:

$$k \equiv (-3)(-6) \equiv 18 \equiv 1 \pmod{17}.$$

Since $k > 1$, we must take $k = 18$. The number of books is then

$$6 + 11 \cdot 18 = 204.$$

PROBLEM 2. Find *all* solutions $x \in \mathbb{Z}$ to the system of congruences

$$x \equiv 2 \pmod 4$$
$$x \equiv 3 \pmod 5$$
$$x \equiv 4 \pmod 9.$$

SOLUTION: To satisfy the first congruence, $x = 2 + 4k$. To satisfy the second, we then have

$$2 + 4k \equiv 3 \pmod 5 \iff 4k \equiv 1 \pmod 5.$$

Since $4 \cdot 4 \equiv 16 \equiv 1 \pmod 4$, multiply through by 4 to get $k \equiv 4 \pmod 5$. So the most general solution to the first two congruences is

$$x = 2 + 4k = 2 + 4(4 + 5\ell) = 18 + 20\ell.$$

To satisfy the last congruence, we have

$$18 + 20\ell \equiv 4 \pmod 9 \iff 2\ell \equiv 4 \pmod 9.$$

The multiplicative inverse of 2 is 5 modulo 9. Therefore,

$$\ell \equiv 5 \cdot 4 \equiv 20 \equiv 2 \pmod{9}.$$

The most general solution to the system is

$$x = 18 + 20(2 + 9t) = 58 + 180t.$$

PROBLEM 3. Find all integers x, y such that

$$\begin{aligned} 2x + 5y &\equiv 4 \pmod{11} \\ x + 3y &\equiv 7 \pmod{11}. \end{aligned}$$

SOLUTION: From the second equation, we have $x \equiv 7 - 3y \pmod{11}$. Substituting into the first equation:

$$4 \equiv 2x + 5y \equiv 2(7 - 3y) + 5y \equiv 14 - y \equiv 3 - y \pmod{11}.$$

Then

$$3 - y \equiv 4 \pmod{11} \iff y \equiv -1 \pmod{11}.$$

So we now have

$$\begin{aligned} x &\equiv 7 - 3y && \pmod{11} \\ y &\equiv -1 && \pmod{11} \end{aligned}$$

or, using the fact that $y \equiv -1 \pmod{11}$,

$$\begin{aligned} x &\equiv 7 - 3(-1) \equiv 10 && \pmod{11} \\ y &\equiv -1 && \pmod{11}. \end{aligned}$$

The most general solution is, thus,

$$x = 10 + 11k \quad \text{and} \quad y = -1 + 11\ell.$$

PROBLEM 4. Does Sunzi's theorem still hold if we drop the requirement that the n_i are relatively prime? Prove your assertion or provide a counterexample.

SOLUTION: The following system, in which $n_1 = n_2 = 2$, is not solvable:

$$\begin{aligned} x &\equiv 0 \pmod{2} \\ x &\equiv 1 \pmod{2}. \end{aligned}$$

Here is another system that has no solutions:

$$x \equiv 1 \pmod 2$$
$$x \equiv 0 \pmod 6.$$

PROBLEM 5. Show there are no integer solutions to the equation

$$x^4 - 125x^3 - 75x^2 + 5x + 15 = 123456789.$$

SOLUTION: If $x \in \mathbb{Z}$ were a solution, then working modulo 5, we would have

$$x^4 - 125x^3 - 75x^2 + 5x + 15 \equiv 123456789 \pmod 5.$$

That equation is equivalent to

$$x^4 \equiv 4 \pmod 5.$$

By Fermat's Little theorem, $x^4 \equiv 1 \pmod 5$ for $x = 1, 2, 3, 4$. The only other possibility is that $x \equiv 0 \pmod 5$, in which case $x^4 \equiv 0 \pmod 5$. Therefore, there are no solutions to $x^4 \equiv 4 \pmod 5$, and hence, there can be no integer solutions to the original equation.