Given a positive integer a, denote by $\overline{a_k a_{k-1} \cdots a_1 a_0}$ its digits in decimal expansion. So in particular, we know that $0 \leq a_i \leq 9$ and

$$a = a_k \cdot 10^k + a_{k-1} \cdot 10^{k-1} + \cdots + a_1 \cdot 10 + a_0.$$

Today we will explore various divisibility tests.

PROBLEM 1. Suppose $c|a-b$. Prove that $c|a$ if and only if $c|b$.

SOLUTION: Since $b = a - (a-b)$, previous work shows that if $c|a-b$ and $c|a$, then $c|b$. The other direction follows similarly from the the fact that $a = b + (a-b)$.

PROBLEM 2.

(a) Prove that a is divisible by 2 if and only if a_0 is divisible by 2.

(b) Prove that a is divisible by 5 if and only if a_0 is divisible by 5.

(c) Prove that a is divisible by 10 if and only if a_0 is 0.

SOLUTION:

(a) Since 2 divides 10, 2 divides 10^i for all $i \geq 1$. Thus 2 divides

$$a_k \cdot 10^k + a_{k-1} \cdot 10^{k-1} + \cdots + a_1 \cdot 10 = a - a_0.$$

Thus 2 divides a if and only if 2 divides a_0.

(b) The proof above works verbatim with 2 replaced by 5.

(c) The proof for (a) works verbatim with 2 replace by 10. Alternatively, a number is divisible by 10 if and only if it is divisible by 2 and 5. Combining the results of parts (a) and (b) then gives this result.

PROBLEM 3.

(a) Describe and prove a rule of divisibility by 4 based on the digits of a.

(b) Describe and prove a rule of divisibility by 8 based on the digits of a.

SOLUTION:

(a) Since 4 divides 100, 4 divides 10^i for all $i \geq 2$. Therefore, 4 divides a if and only if 4 divides $\overline{a_1 a_0}$ following the same logic as above.

(b) Similarly, since 8 divides $1,000$, 8 divides 10^i for al $i \geq 3$. Therefore, 8 divides a if and only if 8 divides $\overline{a_2 a_1 a_0}$.

PROBLEM 4.

(a) Prove that a is divisible by 3 if and only if $\sum_{i=0}^{k} a_i$ is divisible by 3.

(b) Prove that a is divisible by 9 if and only if $\sum_{i=0}^{k} a_i$ is divisible by 9.

SOLUTION:

(a) Note that 3 divides $10^i - 1$ for all $i \geq 0$. Therefore, 3 divides

$$\sum_{i=0}^{k}(10^i - 1)a_i = \left(\sum_{i=0}^{k} 10^i a_i\right) - \left(\sum_{i=0}^{k} a_i\right) = a - \left(\sum_{i=0}^{k} a_i\right).$$

Thus, just as above, 3 divides a if and only if 3 divides $\sum_{i=0}^{k} a_i$.

(b) The proof above works verbatim with 3 replaced by 9.