PROBLEM 1. Today is Monday. What day will it be 3^{20} days from today?

SOLUTION: Notice that $3^{20} = 3^{18} \cdot 3^2 = (3^3)^6 \cdot 3^2$. By Fermat's little theorem, since 7 is prime and does not divide 3^3, we have that 7 divides $(3^3)^6 - 1$. Since divisibility is transitive, we get that 7 divides $(3^3)^6 \cdot 3^2 - 3^2$, that is, there exists k such that $3^{20} - 3^2 = 7k$. This means that 3^{20} and $3^2 = 9$ have the same remainder when dividing by 7, namely 2. Thus, if today is Monday, it will be Wednesday 3^{20} days from today.

PROBLEM 2. As an intrepid wagon wheel painter living in the Olde West, you strive to bring the highest quality, most engaging, non-monochromatic spoke paintings to your customers. You offer wagon wheels with p spokes, where p is a prime integer, painted in up to a colors, where where a is a positive integer.

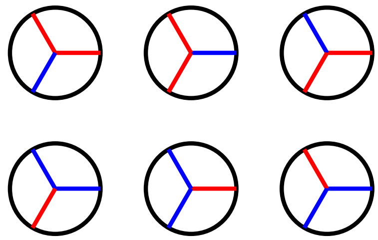
(a) As part of your preparation for painting, you have nailed a wagon wheel to the wall so that it can't rotate. In how many ways can you paint its spokes, assuming that each spoke gets a single color but at least two of the spokes are different colors? (Check your solution in the case $p = 3$ and $a = 2$ by drawing the possibilities.)

(b) When you take the wheel off of the wall and fix it to an axle, you remember that it will rotate, and that your demanding customers will not accept rotated spoke paintings as genuinely different. As you turn this particular wheel around, you notice something remarkable: all of the rotations by multiples of $2\pi/p$ result in distinct colorings in the wheel-nailed-to-wall sense of unique, despite the fact that there are multiple spokes of the same color. Is this a special property of your particular spoke painting, or is it true of all possible non-monochromatic paintings with a colors?

(c) Use your work in (b) to determine the total number of wagon wheel paintings which your customers will accept as genuinely different. What can you deduce from the fact that this number is an integer?

SOLUTION:

(a) If we allow all colorings with each spoke one of a colors, then there are a^p colorings. Of these, a colorings are monochromatic, so there are $a^p - a$ non-monochromatic colorings. Here are the $2^3 - 2 = 6$ colorings for the case $p = 3$ and $a = 2$:



(b) The phenomenon is generic when the number of spokes is prime. Call k a *period* if the coloring is the same after rotating by $2\pi k/p$. Note at few facts: p is a period; if k is a period, then so is $-k$ (rotating in the opposite direction k times); and if k and k' are periods, so is $k+k'$. Now let k be the least period such that $1 \le k < p$. Use the division algorithm to write $p = kq + r$ with $0 \le r < p$. Since both p and k are periods, so is r. Given the minimality of k, this means that $r = 0$. Hence, k divides p. Since p is prime, this forces $k = 1$ pr $k = p$. If $k = 1$, the pattern is monochromatic.

(c) The nailed-to-the-wall count of $a^p - a$ overcounts by a factor of p (the number of ways to rotate one pattern into others). Thus $\frac{a^p-a}{p}$ is an integer; in particular, p divides $a^p - a$. This is *Fermat's little theorem*.