PROBLEM 1. The digits 1, 2, 3, 4 are randomly arranged into two two-digit numbers $\overline{AB}$ and $\overline{CD}$—each of the four digits is used exactly once. In this problem you will ultimately determine the expected value of $\overline{AB} \cdot \overline{CD}$.

(a) Randomly choose two digits from the set $\{1, 2, 3, 4\}$ without replacement (for example, we cannot choose 1 twice). What is their expected product? [To get started: create an appropriate sample space S and random variable $X\colon S \to \mathbb{R}$.]

(b) Note that $\overline{AB}$ is a linear combination of A and B: namely, $\overline{AB} = 10A + B$. A similar statement holds for $\overline{CD}$. Use this fact along with part (i) and linearity of expectation to determine the expected value $E(\overline{AB} \cdot \overline{CD})$.

(c) (Return to this after you are done with problem 2.) What is the expected value of drawing a single number from $\{1, 2, 3, 4\}$? Given your answer to part (a), is the expected value multiplicative?

PROBLEM 2. (The coupon collector problem.) Safeway is running a promotion in which they have produced n coupons and you randomly receive a coupon each time you check out. You passionately hope to one day collect all n coupons. What is the expected number of times T you'll have to check out at the store in order to collect all n? There's a very clever way to solve this problem with linearity of expectation!

(a) Label the coupons $C_1, C_2, \ldots, C_n$. If $n = 4$, a successful collection of all 4 coupons might look like $C_2\, C_2\, C_4\, C_2\, C_1\, C_3$. Break the sequence into segments where a segment ends when you receive a new coupon. In the example sequence, the segments are:

$$C_2, \quad C_2\, C_4, \quad C_2\, C_1, \quad C_3.$$

Because it will make our lives easier, consider these the 0-th, 1-st, ..., 3-rd segments (as opposed to 1-st through 4-th). Let X_k be the length of the k-th segment, and note that k ranges from 0 through $n-1$. In the example, $X_0 = 1$, $X_1 = 2$, $X_2 = 2$, and $X_3 = 1$. Express T, the total number of checkouts needed to collect all coupons, as a linear combination of the X_k.

(b) Compute p_k, the probability that you will collect a new coupon given that you have already collected k of them. After studying the geometric distribution, we will learn that $E(X_k) = 1/p_k$. Compute this value.

(c) Use your answers to (i) and (ii) to determine $E(T)$.

(d) Can you say anything about the asymptotic behavior of $E(T)$?