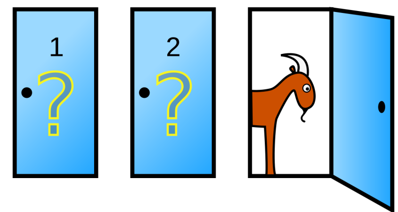
PROBLEM 1 (The Monty Hall problem). A game show provides contestants with the opportunity to win a car. There are three doors labeled A, B, and C. Behind two of the doors are goats, and behind one of the doors is a car. For reasons not completely clear to your instructor, you hope to select the car instead of a goat. The game proceeds in the following fashion: First, you select a door. Next, the host reveals a goat behind one of the remaining doors. (Since there are two goats, there is at least one goat to reveal.) You are then given the chance to switch your guess. If your final guess is the door with the car behind it, you win the car. **Question:** Is it advantageous to switch your guess?



Here are some assumptions on the problem which should remove any ambiguity:

- The probability that the car is placed behind any one of the three doors is $1/3$.
- The host knows where the car is.
- If the contestant picks a door with a goat behind it at the beginning, the host opens the remaining door with a goat before giving the option to switch. If the contestant picks the door with the car behind it, the host opens any of the other doors with probability $1/2$.

Suppose that you initially pick door A and then let A, B, and C denote the events "the car is behind door A," "door B," and "door C," respectively. Let M_A, M_B, and M_C denote the events "the host opens door A," "door B," and "door C," respectively.

(a) What are $P(M_C|A)$, $P(M_C|B)$, and $P(M_C|C)$?

(b) What is $P(M_C)$? (Use the Law of Total Probability.)

(c) Suppose that the host opens door C revealing a goat. You should switch your guess to B if $P(B|M_C) > P(A|M_C)$. Compute these conditional probabilities (via Bayes' Law) and draw a conclusion.

SOLUTION:

(a) If the car is behind door A, then the rules say that the host chooses between B and C with equal probability. So $P(M_C|A) = 1/2$. If the car is behind door B, then since the host does not want to reveal the car and A is already chosen, Monty's only choice is C. So $P(M_C|B) = 1$. Finally, since the host cannot reveal the car, $P(M_C|C) = 0$. To summarize:

$$P(M_C|A) = 1/2, \quad P(M_C|B) = 1, \quad \text{and} \quad P(M_C|C) = 0.$$

(b) We have

$$
\begin{aligned}
P(M_C) &= P(M_C|A)P(A) + P(M_C|B)P(B) + P(M_C|C)P(C) \\
&= \frac{1}{2} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} \\
&= \frac{1}{2}.
\end{aligned}
$$

(c) Finally, by Bayes' law,

$$
\begin{aligned}
P(A|M_C) &= \frac{P(M_C|A)P(A)}{P(M_C)} \\
&= \frac{1/2 \cdot 1/3}{1/2} \\
&= \frac{1}{3}.
\end{aligned}
$$

and

$$
\begin{aligned}
P(B|M_C) &= \frac{P(M_C|B)P(B)}{P(M_C)} = \frac{1 \cdot 1/3}{1/2} \\
&= \frac{2}{3}.
\end{aligned}
$$

The latter quantity is twice as large as the first, so we should switch!

PROBLEM 2. Angélica is taking a true-false test and always marks the correct answer when she knows it and decides true or false on the basis of flipping a fair coin when she does not know it. If the probability that she will know an answer is 3/5, what is the probability that she knew the answer to a correctly marked question?

SOLUTION: Let K denote the event of knowing the answer to a particular problem and let C denote the event of correctly marking that problem. We want to determine $P(K|C)$, and do so with Bayes' Law. First note that the problem tells us that $P(K) = 3/5$, $P(C|K) = 1$, and $P(C|K^c) = 1/2$. (Here K^c is the complement of K, the event in which the student does not know the answer.) By the Law of Total Probability,

$$P(C) = P(C|K)P(K) + P(C|K^c)P(K^c) = 1 \cdot 3/5 + 1/2 \cdot 2/5 = \frac{4}{5}.$$

Thus

$$P(K|C) = \frac{P(C|K)P(K)}{P(C)} = \frac{1 \cdot 3/5}{4/5} = \frac{3}{4}.$$

In other words, there is a 75% chance of Angélica knowing the answer to a correctly marked question.