Let S be our sample space (really any set) and let 2^S denote the corresponding collection of events (just the set of subsets of S). Recall that a *probability distribution* on S is a function

$$P : 2^S \to [0,1]$$

such that (1) $P(S) = 1$, (2) $P(\varnothing) = 0$, and (3) if $A, B \in 2^S$ are mutually exclusive events (so $A \cap B = \varnothing$), then $P(A \cup B) = P(A) + P(B)$. If S is a finite set, then we can define the *uniform probability distribution* on S to be the function taking $A \subseteq S$ to $|A|/|S|$.

PROBLEM 1. A lottery has participants choose 5 distinct numbers from the set $[36] = \{1, 2, \ldots, 36\}$. On a prescribed date, the lottery announces a collection of 5 winning numbers, i.e., a subset of [36] of cardinality 5. Complete the following prompts in order to determine why the lottery does not offer a prize for having selected just 1 or more winning numbers.

(a) What sample space is pertinent in this question? Describe it both as a collection of certain types of objects, and in a more mathematical fashion.

(b) Is it reasonable to put the uniform probability distribution on this sample space? (Assume that the lottery is fair.)

(c) Let B denote the event of choosing a ticket with no winning numbers. What is $P(B)$?

(d) Let A denote the event of choosing a ticket with at least one winning number. What is $A \cap B$? $A \cup B$?

(e) Use the axioms for a probability distribution and your answer to (iii) to determine $P(A)$.

(f) [Follow up question] Might it be reasonable to offer prizes for anyone with 2 or more winning numbers?

SOLUTION:

(a) The sample space is the collection of valid lottery tickets. If we assume that the lottery does not care about the order of the numbers, then we may model this sample space as $\binom{[36]}{5}$, the collection of 5-element subsets of $[36] = \{1, 2, \ldots, 36\}$.

(b) Sure! If the lottery is fair, then each ticket has an equal chance of being drawn.

(c) Suppose the winning ticket is the set $\{a_1, a_2, a_3, a_4, a_5\}$ where the a_i are distinct elements of [36]. Then $B = \{t \in \binom{[36]}{5} \mid a_i \notin t\}$. In other words, B is the collection of 5-element subsets of $[36] \smallsetminus \{a_1, \ldots, a_5\}$. As such $|B| = \binom{31}{5}$ and $P(B) = \binom{31}{5}/\binom{36}{5} \approx 0.45$.

(d) We have $A \cap B = \varnothing$ and $A \cup B = \binom{[36]}{5}$.

(e) It follows that $P(A) = P(A \cup B) - P(B) = 1 - \binom{31}{5}/\binom{36}{5} \approx 0.55$. If the lottery pays out 55% of the time, then it's not a very lucrative lottery for those running it!

(f) The number of ways of choosing a ticket with no winning numbers is $\binom{31}{5}$, and the number of way of choose a ticket with exactly one winning number is $5 \cdot \binom{31}{4}$. For the latter, there are 5 ways of choosing the winning number, and then $\binom{31}{4}$ ways to choose the remaining 4 numbers from among the 31 non-winners. Thus, the probability of receiving 2 or more winning numbers is

$$1 - \frac{\binom{31}{5} + 5 \cdot \binom{31}{4}}{\binom{36}{5}} \approx 0.13.$$

That's at least more reasonable since $0.13 < 0.5$.

PROBLEM 2. What is the probability that in a random ordering of a standard deck of cards (which has 52 cards), the ace of spades precedes the king of hearts?

(a) Rephrase this as a question about permutations of [52]. What is the sample space under consideration? the event?

(b) Prove that the probability of this event (under the uniform distribution) is 1/2 by producing a bijection between the event and its complement. (Why does that solve things?)

SOLUTION:

(a) We can number the cards 1 through 52, designating the ace of spades 1 and the king of hearts 2. An ordering of the cards corresponds to a permutation of [52], so the sample space is $\mathfrak{S}_{52}$, the set of permutations of [52]. The event is

$$A = \{\pi \in \mathfrak{S}_{52} \mid \pi(1) < \pi(2)\}.$$

(b) We have $\mathfrak{S}_{52} \smallsetminus A = \{\pi \in \mathfrak{S}_{52} \mid \pi(2) < \pi(1)\}$. This is in bijection with A via the function that swaps the values of $\pi(1)$ and $\pi(2)$. Thus $|A| = |B|$, $A \cup B = \mathfrak{S}_{52}$, and $A \cap B = \varnothing$. As such,

$$1 = P(A \cup B) = P(A) + P(B) = 2P(A)$$

whence $P(A) = 1/2$.

Challenge

Challenge problems are optional and should only be attempted after completing the previous problems.

Your friend invites you to play a game: they write ten distinct real numbers on ten blank cards. The cards are shuffled randomly and placed face down on the table. You start at the top of the deck and start revealing cards. At any point you may choose to stop turning over cards and select the most recently revealed card. You win if your selection is the largest of all ten numbers (both those previously revealed and those still unrevealed). Devise a strategy which guarantees you will win this game at least 25% of the time.

SOLUTION: This is a variant on the so-called *secretary problem*, née *fianceé problem*. We can use a *stopping rule* to increase our chance of winning: look at the first r cards and note the maximal value among them, M. For the subsequent $10 - r$ cards, select the first one larger than M. (If the tenth is not larger than M, select it and and bemoan your bad luck). With $r = 4$, you will select the largest number about 40% of the time, and this is the best r for 10 cards. A full analysis can be found in (Sardelis and Valahas, *Decision Making: A Golden Rule*, The American Mathematical Monthly Vol. 106, No. 3 (Mar., 1999), pp. 215-226).