PROBLEM 1. Let $a \neq b$ and consider the expression

$$\frac{x}{(1-ax)(1-bx)}.$$

Check that

$$\frac{x}{(1-ax)(1-bx)} = \frac{1}{a-b}\left(\frac{1}{1-ax} - \frac{1}{1-bx}\right).$$

SOLUTION: We will present a solution to the following problem, which subsumes the one above but doesn't assume we know the right-hand side. Find C and D such that

$$\frac{x}{(1-ax)(1-bx)} = \frac{C}{1-ax} + \frac{D}{1-bx}.$$

Multiplying both sides by $(1-ax)(1-bx)$ yields

$$x = C(1-bx) + D(1-ax) = (C+D) - (Cb+Da)x.$$

Since this is an equality of polynomials, it implies that

$$0 = C + D \quad \text{and} \quad 1 = -(Cb + D).$$

Solving for C and D we get

$$C = \frac{1}{a-b} \quad \text{and} \quad D = \frac{1}{b-a}.$$

PROBLEM 2. Consider the sequence $a_0, a_1, \ldots$ defined by the recurrence

$$a_0 = 0, \quad a_1 = 1, \quad \text{and} \quad a_n = 5a_{n-1} - 6a_{n-2} \text{ for } n \geq 2.$$

(a) Write out the terms of (a_n) until you get to 2059.

(b) Lex $A(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots$ be the generating function for the sequence $(a_n)_{n=0}^{\infty}$. In the text, we used the Fibonacci recurrence to find a closed expression for the generating function. Apply a similar procedure to $A(x)$.

(c) Use part (ii) to find a closed form for (a_n).

SOLUTION:

(a) $0, 1, 5, 19, 65, 211, 665, 2059.$

(b) We have

$$\begin{aligned} A(x) &= x + 5x^2 + 19x^3 + 65x^4 + 211x^5 + \cdots + \quad a_n x^n + \cdots, \\ 5xA(x) &= \quad + 5x^2 + 25x^3 + 95x^4 + 325x^5 + \cdots + 5a_{n-1}x^n + \cdots, \\ 6x^2A(x) &= \quad + \quad + \; 6x^3 + 30x^4 + 114x^5 + \cdots + 6a_{n-2}x^n + \cdots. \end{aligned}$$

It follows from the recurrence that $A(x) - x = 5xA(x) - 6x^2A(x)$. Solving for $A(x)$ gives

$$A(x) = \frac{x}{1 - 5x + 6x^2}.$$

(c) Using part (ii), we have

$$\begin{aligned} A(x) &= \frac{x}{1 - 5x + 6x^2} \\ &= \frac{x}{(1-3x)(1-2x)} \\ &= \frac{1}{3-2}\left(\frac{1}{1-3x} - \frac{1}{1-2x}\right) \\ &= \frac{1}{1-3x} - \frac{1}{1-2x} \\ &= \sum_{n\geq 0} 3^n x^n - \sum_{n\geq} 2^n x^n \\ &= \sum_{n\geq 0} (3^n - 2^n) x^n. \end{aligned}$$

It follows that $a_n = 3^n - 2^n$.

PROBLEM 3. Let $f(x) = \sum_{i=0}^{\infty} b_i x^i$ be the generating function for the sequence $b_0, b_1, \ldots$.

(a) Let $g(x) = (1-x)f(x)$. Then $(g(x) - b_0)/x$ is the generating function for which sequence?

(b) Let $h(x) = \frac{f(x)}{1-x}$. Then $h(x)$ is the generating function for which sequence?

(c) Apply the previous result to $h(x) = 1/(1-x)$ to find the sequence whose generating function is $1/(1-x)^2$.

(d) Find the sequence whose generating function has closed form $\frac{1+x+x^2}{(1-x)^2}$ by multiplying $1 + x + x^2$ by the series for $1/(1-x)^2$.

SOLUTION:

(a)

$g(x) = (1-x)f(x)$

$$
\begin{aligned}
&= (1-x)(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots) \\
&= (b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots) - x(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots) \\
&= (b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots) - (b_0 x + b_1 x^2 + b_2 x^2 + b_3 x^4 + \cdots) \\
&= b_0 + (b_1 - b_0)x + (b_2 - b_1)x^2 + (b_3 - b_2)x^3 + \cdots \\
&= b_0 + \Delta[b]_0 x + \Delta[b]_1 x^2 + \Delta[b]_2 x^3 + \cdots .
\end{aligned}
$$

Therefore,

$$\frac{g(x) - b_0}{x}$$

is the generating function for $\Delta[b]$.

(b)

$$
\begin{aligned}
\frac{f(x)}{1-x} &= \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots\right)\left(1 + x + x^2 + x^3 + \cdots\right) \\
&= b_0 + (b_0 + b_1)x + (b_0 + b_1 + b_2)x^2 + (b_0 + b_1 + b_2 + b_3)x^3 + \cdots
\end{aligned}
$$

So $f(x)/(1-x)$ is the generating function for the sequence whose n-term is $\sum_{i=0}^{n} b_i$.

(c) We get

$$\frac{1}{(1-x)^2} = \frac{1 + x + x^2 + x^3 + \cdots}{1-x} = 1 + 2x + 3x^2 + 4x^3 + \cdots .$$

(d) We have

$$
\begin{aligned}
\frac{1 + x + x^2}{(1-x)^2} &= (1 + x + x^2)(1 + 2x + 3x^2 + 4x^3 + \cdots) \\
&= (1 + 2x + 3x^2 + 4x^3 + \cdots) + x(1 + 2x + 3x^2 + 4x^3 + \cdots) + x^2(1 + 2x + 3x^2 + 4x^3 + \cdots) \\
&= 1 + 3x + 6x^2 + 9x^3 + \cdots .
\end{aligned}
$$

For $n \geq 1$, the coefficient for x^n is $(n+1) + n + (n-1) = 3n$. The sequence is (a_n) where $a_0 = 1$ and $a_n = 3n$ for $n > 1$.