PROBLEM 1. Suppose that $a \in \mathbb{R}^{\mathbb{N}}$ is a polynomial sequence of degree 4. Use the following table of differences to determine a formula for a_n.

a_n	0		0		4		12		72			...
$\Delta[a]_n$		0		4		8		60			...	
$\Delta^2[a]_n$			4		4		52			...		
$\Delta^3[a]_n$				0		48		...				
$\Delta^4[a]_n$					48		...					

SOLUTION: We have

$$a_n = 48\binom{n}{4} + 4\binom{n}{2} = 2n^4 - 12n^3 + 24n^2 - 14n.$$

PROBLEM 2. With your group, choose a "random" polynomial p of degree at most 3. Prepare a table of the values $p(n)$ for $n = 0, 1, 2, 3$. Swap tables of values with another group and then reconstruct each others polynomials. Here is an example of https://sagecell.sagemath.org/ code that might help:

```
f(x) = 5*x^3 + 2*x^2 + 10*x + 6
[f(i) for i in range(5)]
```

Output:

```
[6, 23, 74, 189]
```

PROBLEM 3.

(a) Fix $r \geq 0$. For $n \geq 0$ define the sequence whose n-th term is $a_n = \sum_{k=0}^{n} k^r$. Prove that $(a_n)_{n=0}^{\infty}$ is a degree $r+1$ polynomial sequence. (Hint: our reading implies that is suffices to show that $\Delta[a]_n$ is a polynomial sequence of degree r.)

(b) Consider the case $r = 3$. Use a table of differences to determine a polynomial expression for

$$a_n = \sum_{k=0}^{n} k^3.$$

Fun fact: you can write that polynomial as the square of a single binomial coefficient involving n.

SOLUTION:

(a) We have $\Delta[a]_n = (n+1)^r$, so a_n has a degree $r+1$ polynomial expression.

(b) The table of differences for a_n in this case is as follows:

a_n	0		1		9		36		100		$\cdots$		
$\Delta[a]_n$		1		8		27		64		$\cdots$			
$\Delta^2[a]_n$			7		19		37		$\cdots$				
$\Delta^3[a]_n$				12		18		$\cdots$					
$\Delta^4[a]_n$					6		$\cdots$						

Thus

$$\begin{aligned} a_n &= 6\binom{n}{4} + 12\binom{n}{3} + 7\binom{n}{2} + \binom{n}{1} \\ &= \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} \\ &= \left(\frac{n(n+1)}{2}\right)^2. \end{aligned}$$

PROBLEM 4. Prove that 3 divides $n^3 + 2n$ for all $n \in \mathbb{N}$ by using a table of differences to write $\frac{1}{3}n^3 + \frac{2}{3}n$ as a sum of $\binom{n}{k}$s.

SOLUTION: We can use a table of differences to express $\frac{1}{3}n^3 + \frac{2}{3}n$ as

$$2\binom{n}{3} + 2\binom{n}{2} + \binom{n}{1}.$$