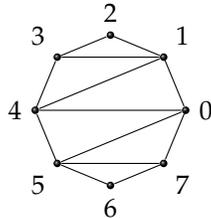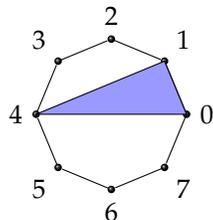PROBLEM 1. A *triangulation* of a convex n-gon is a collection of non-intersecting diagonals (line segments between non-adjacent vertices) that break the n-gon into triangles. For instance, the following is a triangulation of an 8-gon:


(a) Let T_n denote the number of triangulations of an n-gon. Draw all triangulations of convex n-gons for $n = 3, 4, 5, 6$, and make a conjecture for the value of T_n, in general.

(b) Prove your conjecture. *Hint*: Label the vertices of the n-gon by $0, 1, \cdots, n-1$, cyclically. Given any triangulation, exactly one triangle will contain the edge 01. Use this triangle as the basis for a recursion. For example, for each $k = 2, 3, \ldots, 7$, how many triangulations are there of the 8-gon which contain the triangle $01k$? For $k = 4$, we would be asking how many ways are there of completing the triangulation of the figure below:

SOLUTION:

(a) There is just one triangulation of $\triangle$, consisting of no diagonals. The following pictures for $n = 4, 5, 6$ are from https://robertdickau.com/catalan.html.

4 sides, 2 ways:
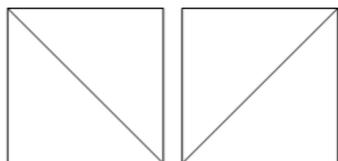
5 sides, 5 ways:

6 sides, 14 ways:

(b) We conjecture that there are C_{n-2} triangulations of a convex n-gon, i.e., that $T_n = C_{n-2}$, for $n \geq 3$. We prove this by induction. For the base case, we note that that $T_3 = 1 = C_1$. For induction, fix $n \geq 3$ and suppose that $T_k = C_{k-2}$ for $k \leq n$. Given a convex $(n+1)$-gon, label the vertices cyclically $0, 1, 2, \ldots, n$. Following the hint, let 01 be the base. Let X_k be the triangulations containing the triangle $01k$. The rest of the triangulation triangulates the convex polygons with vertices $1, 2, \ldots k$ and with vertices $k, k+1, \ldots, n, 0$. These are a k-gon and $(n-k+2)$-gon, respectively. Thus

$$|X_k| = T_k T_{n-k+2}.$$

Clearly $X_2 \amalg X_3 \amalg \cdots \amalg X_n$ is a partition of all triangulations of the $(n+1)$-gon, so

$$
\begin{aligned}
T_{n+1} &= \sum_{k=2}^{n} T_k T_{n-k+2} \\
&= \sum_{k=2}^{n} C_{k-2} C_{n-k} \\
&= C_0 C_{n-2} + C_1 C_{n-3} + \cdots + C_{n-2} C_0 \\
&= C_{n-1}
\end{aligned}
$$

by Proposition 116.

PROBLEM 2. A *binary necklace* is a collection of blue and yellow beads strung along a circle. We count to necklaces as being the same if one can be obtained from the other by sliding the beads. Thus, the two

necklaces in Figure 1 are the same. However, when you are comparing necklaces to see if they are the same, you are *not* allowed to flip them over.
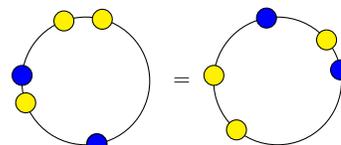

Figure 1: Two views of the same necklace.

(a) For $n = 0, 1, 2, 3, 4$, count the number of binary necklaces with n blue beads and $n + 1$ yellow beads.

(b) Find a formula that counts the number of binary necklaces with n blue beads and $n + 1$ yellow beads, using binomial coefficients.

(c) Compare to the formula for Catalan numbers.

SOLUTION:

(a) We find that the values for $n = 0, 1, 2, 3, 4$ are $1, 1, 2, 5, 14$.

(b) Consider the $2n + 1$ beads as fixed in position. If we don't count rotations as the same, there are $\binom{2n+1}{n}$ necklaces, corresponding to the ways of choosing the positions of the n blue beads. Now given a fixed arrangement, there are $2n + 1$ possible rotations, and they are all distinct. The latter follows from the fact that if there were two rotations giving the same distribution, it would imply there is a pattern of beads that repeats itself, and the number of times it repeats must divide n and $n + 1$. Since n and $n + 1$ don't have any common factors, there can't be any repetitions. This means then that each necklace appears $2n + 1$ times in the original count, giving us a final answer of

$$\frac{1}{2n + 1} \binom{2n + 1}{n}.$$

(c) Using Theorem 134, we have the following

$$
\begin{aligned}
C_n &= \frac{1}{n + 1} \binom{2n}{n} \\
&= \frac{1}{n + 1} \cdot \frac{(2n)!}{n! \cdot n!} \\
&= \frac{(2n)!}{(n + 1)! \cdot n!} \\
&= \frac{2n + 1}{2n + 1} \cdot \frac{(2n)!}{(n + 1)! \cdot n!} \\
&= \frac{1}{2n + 1} \cdot \frac{(2n + 1)!}{(n + 1)! \cdot n!} \\
&= \frac{1}{2n + 1} \binom{2n + 1}{n}.
\end{aligned}
$$

We thus have that the number of binary necklaces with n blue bids and $n + 1$ yellow bids is C_n.

PROBLEM 3. From our reading, we know that full binary trees with $n + 1$ leaves and balanced parenthesizations of length $2n$ are counted by the Catalan number C_n. The reading also includes a description of a direct bijection between these two structures. Briefly, given a full binary tree, label the left edges with '(' on their left and ')' on their right. Start at the root of the tree and start walking down the leftwards edge; keep the tree on your left and record the labels as you pass them. The resulting is the balanced parenthesization corresponding to the binary tree.

   Prove that the process described above works, i.e., that it provides a bijection. It is recommended that you follow these steps:

(a) Draw several full binary trees and produce the resulting balanced parenthesizations.

(b) Why is the resulting parenthesization always balanced?

(c) Now create several balanced parenthesizations, and find their corresponding binary trees.

(d) Describe an algorithm (or function) for turning a balanced parenthesization into a full binary tree which is inverse to our process of turning binary trees into balanced parenthesizations.

SOLUTION: We jump straight to the proof. Observe that the parenthesization is balanced because the ')' to the right of an edge always closes the '(' on the left. Given a balanced parenthesization, create a full binary tree by starting at the root and drawing two child nodes for the opening (and moving down and left. For each subsequent (draw two children and move down and left. When you meet a), go to the nearest leaf which you haven't visited already. Continue following these instructions until you exhaust the parenthesization. The procedure only adds two children at a time, so we necessarily produce a full binary tree. During the procedure, we visit all but the rightmost node with each), so there are $n + 1$ leaves in the tree. It is now straightforward to verify that these functions are inverses of each other.

Challenge

   Produce a direct bijection between triangulations of a convex n-gon and full binary trees with $n - 1$ leaves. Show that diagonal flips of edges in a triangulation correspond to tree "rotations". (A *diagonal flip* transforms a pair of edge-sharing triangles from ◹ into ◺.) [Hint: Identify the vertices of the tree with the edges of the triangulation, including the edges of the n-gon. Fix one of the edges of the n-gon to be the root of the tree. Which two vertices (edges of the triangulation)

should be adjacent to the root? How can you continue to grow the tree from there?] Carry out the procedure for the triangulation of the octagon given at the start of this problem set. What happens to the binary tree if you flip the triangulation of the quadrilateral 0134 in that example?

SOLUTION: Using the hint, the vertices of the tree will be the edges of the triangulation (including the edges of the polygon). Label one of the polygon edges as the root. Draw edges in the tree to vertices corresponding to the other two edges in triangulation. Leaves will correspond to the non-root edges of the polygon, of which there are $n - 1$. We leave the remaining details to the reader.

For example, consider the triangulation from above:
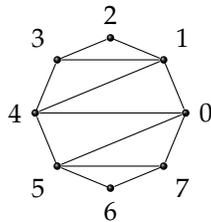
The triangulation has 13 edges, and so our tree will, too. We label the root of the tree as 01. Since 01 is in the triangle 014, our tree starts
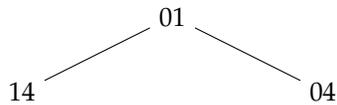
Note that we had to choose a sideness, we chose the ones that matches how the three edges of the triangle relate to each other. We now grow our tree from the newly created vertices 14 and 04. Besides the already-considered triangle 014, the edge 14 is contained in exactly one other triangle: 134. Besides 04, that triangle contains the edges 05 and 45. These will be the next vertices to add to our tree:
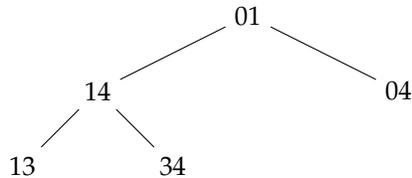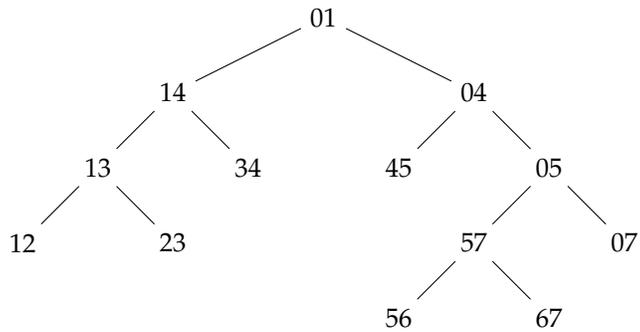

The next step will be to consider the vertex 04. Besides triangle 014, it is in triangle 045, so we continue to grow our tree:
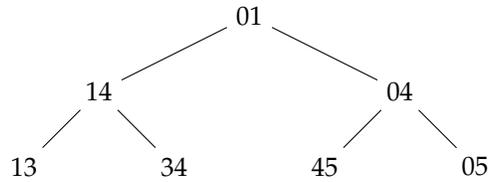
We continue growing from the leaves of this tree until we can go no further. The leaves of the final tree will correspond to the $n-1$ edges of the 8-gon, not including 01.

In this particular example, we eventually get

Flipping the triangulation of 0134 has the effect of switching the vertex labeled 14 to one labeled 03 that is on the other side of 01. This vertex is still connected to 34. We "rotate" 14 up to towards 01, and when it comes down to the right of it, it changes label to 3. This is the result: