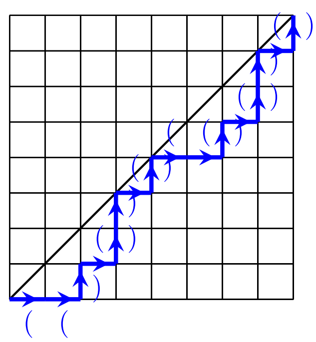
PROBLEM 1. In your reading, you saw that expressions consisting of n balanced parentheses () are in bijection with Dyck paths of length $2n$, and, thus, the number of such expressions is the n-th Catalan number, C_n.

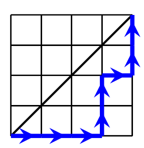
SOLUTION:

(a) Describe the bijection between Dyck paths and balanced parentheses, and apply it to the Dyck path below.



$\longmapsto$ (()())()(()())().

(b) What is the Dyck path associated with ((())())?



PROBLEM 2. Use the Catalan recurrence,

$$C_0 = 1 \quad \text{and} \quad C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k} \text{ for } n \geq 0,$$

to compute C_1 then C_2, then, etc, up to C_5.

SOLUTION: We have

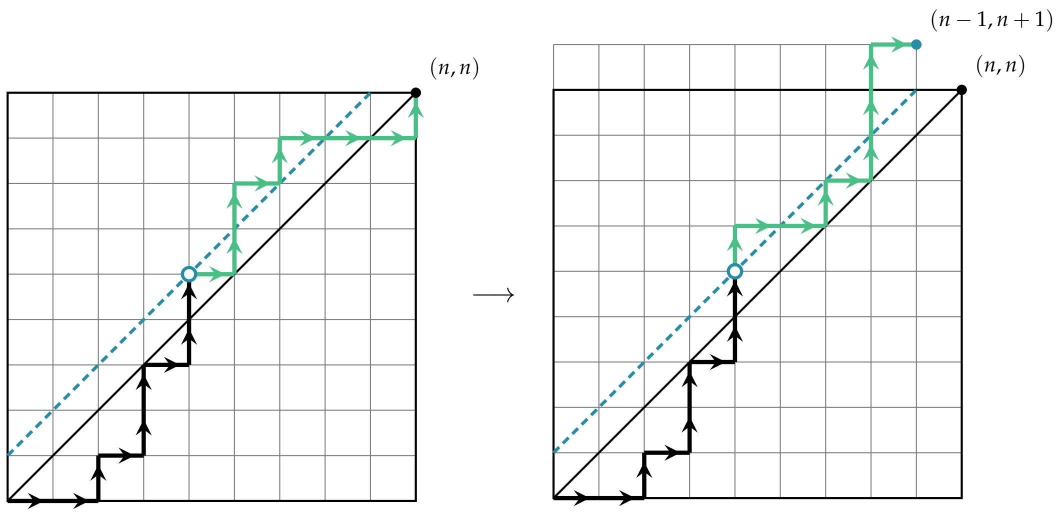
$$C_1 = C_0C_0 = 1$$
$$C_2 = C_0C_1 + C_1C_0 = 1 + 1 = 2$$
$$C_3 = C_0C_2 + C_1C_1 + C_2C_0 = 2 + 1 + 2 = 5$$
$$C_4 = C_0C_3 + C_1C_2 + C_2C_1 + C_3C_0 = 5 + 2 + 2 + 5 = 14$$
$$C_5 = C_0C_4 + C_1C_3 + C_2C_2 + C_3C_1 + C_4C_0 = 14 + 5 + 4 + 5 + 14 = 42.$$

PROBLEM 3. We will give a combinatorial proof that

$$C_n = \binom{2n}{n} - \binom{2n}{n+1}.$$

Recall that C_n is the number of Dyck paths of length $2n$, and $\binom{2n}{n}$ is the number of all NE paths from $(0,0)$ to (n,n). Let's call a NE path that travels above the main diagonal *cavalier*. Thus, it suffices to prove that the number of cavalier paths from $(0,0)$ to (n,n) is $\binom{2n}{n+1}$.

Let C denote the set of cavalier paths from $(0,0)$ to (n,n), and P the set of NE paths from $(0,0)$ to $(n-1,n+1)$. Consider the following construction. Given a cavalier path, mark the first time it crosses the main diagonal and touches the next higher diagonal (dashed blue in the example below). Then reflect the remainder of the path (in green in the example below) across that blue dashed diagonal, so that east steps become north, and north steps become east.



(a) Illustrate this construction for all the cavalier paths when $n = 1$ and $n = 2$.

(b) Convince yourself that this construction gives a well-defined function

$$f : C \to P.$$

This means making sure of two things:

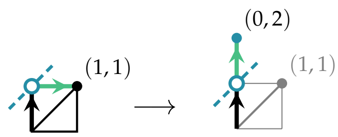
- This construction is uniquely defined—given a cavalier path p, there is exactly one way to construct $f(p)$; and
- the result is indeed a NE path from $(0,0)$ to $(n-1,n+1)$.

(This is not trivial to do.)

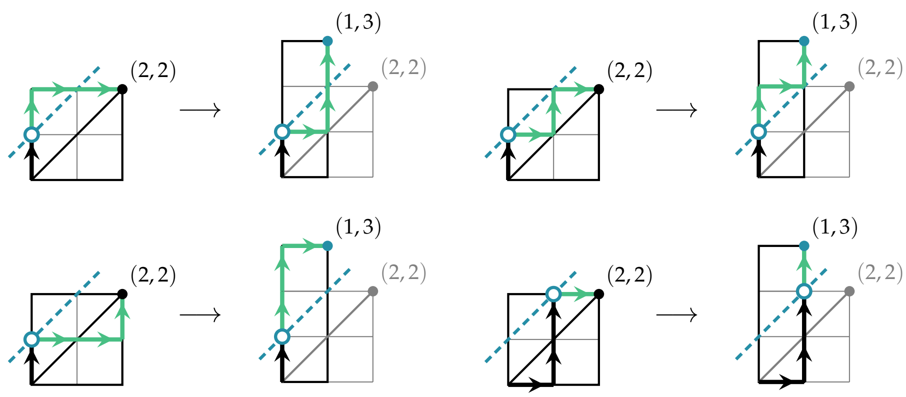
(c) Prove that f is a bijection.

SOLUTION:

(a) $n = 1$:



$n = 2$:

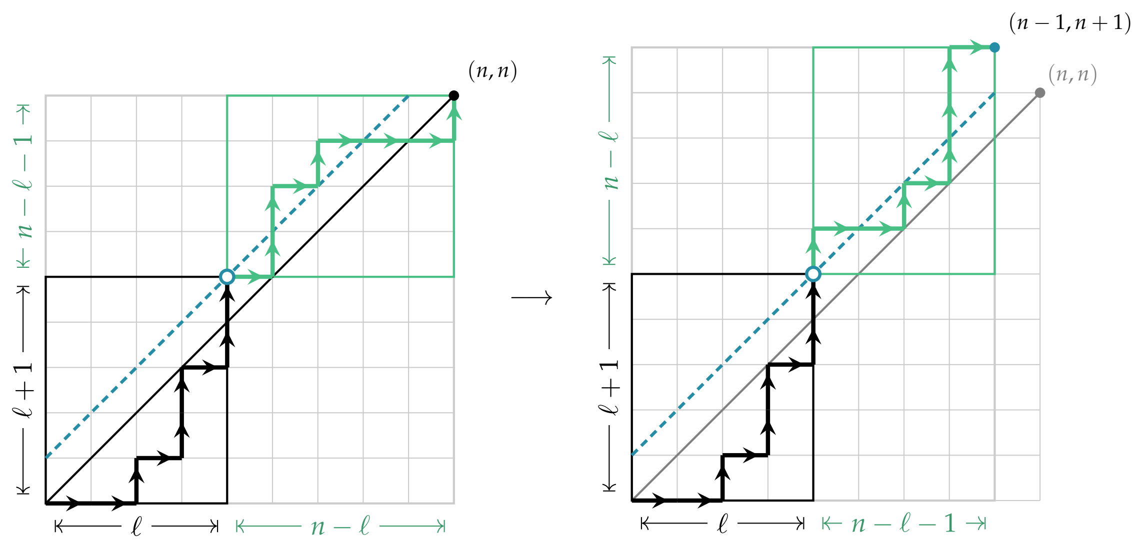


(b) Let p be a cavalier path. Since p travels strictly above the main $y = x$ diagonal, it touches the dashed diagonal at least once. The first time this happens coincides with the first time there are more N's than E's; in particular, at that point, there is *exactly* one more N than E's. Suppose, at that point, the path has accumulated ℓ E's (and hence $\ell + 1$ N's). Then the remaining (green) portion of the path has exactly

$$n - \ell \quad \text{E's} \qquad \text{and} \qquad n - \ell - 1 \quad \text{N's}.$$

Hence, flipping N's and E's after that point will result in a total

$$\begin{aligned} \ell + (n - \ell - 1) &= n - 1 \quad \text{E's,} \quad \text{and} \\ \ell + 1 + (n - \ell) &= n + 1 \quad \text{N's.} \end{aligned}$$



So $f(p)$ is a NE lattice path from $(0,0)$ to $(n-1, n+1)$. [Alternatively, since the reflection across the blue diagonal sends (n,n) to $(n-1, n+1)$, you know that is where the path will end.]

(c) Any NE path from $(0,0)$ to $(n-1, n+1)$ must touch the dashed diagonal at some point (since it ends above the main $y = x$ diagonal at the end). Thus we can define the inverse function from P to C by reflecting the remainder of the path after it touches the dashed diagonal for the first time.