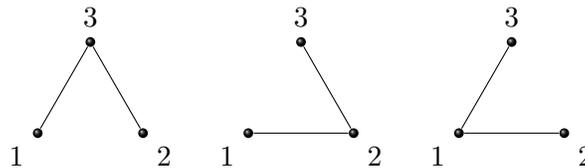


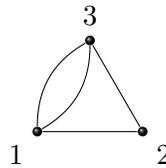
**MATH 113: DISCRETE STRUCTURES  
HOMEWORK 17**

**Due:** Monday, March 9 at 10pm.

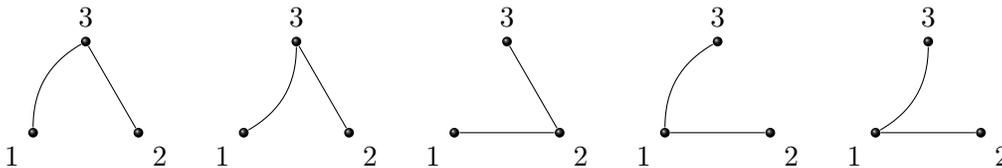
A *spanning tree* of a connected graph  $G$  is a subgraph  $T$  such that  $T$  is a tree and every vertex of  $G$  is on some edge of  $T$ . For instance, if  $G$  is the triangle with vertices 1, 2, 3, then its spanning trees are:



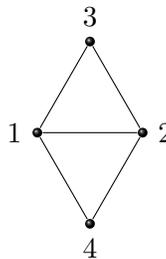
Recall that a *multigraph* is a graph in which multiple edges are allowed. For instance, the following graph has two edges connecting the vertices 1 and 3:



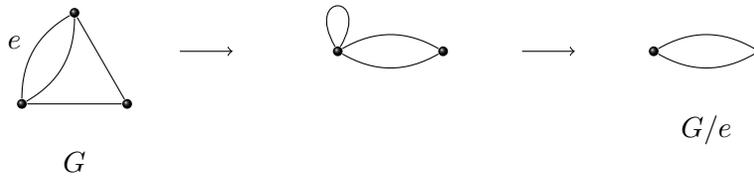
It has five spanning trees:



*Problem 1.* Draw all spanning trees of the following graph:



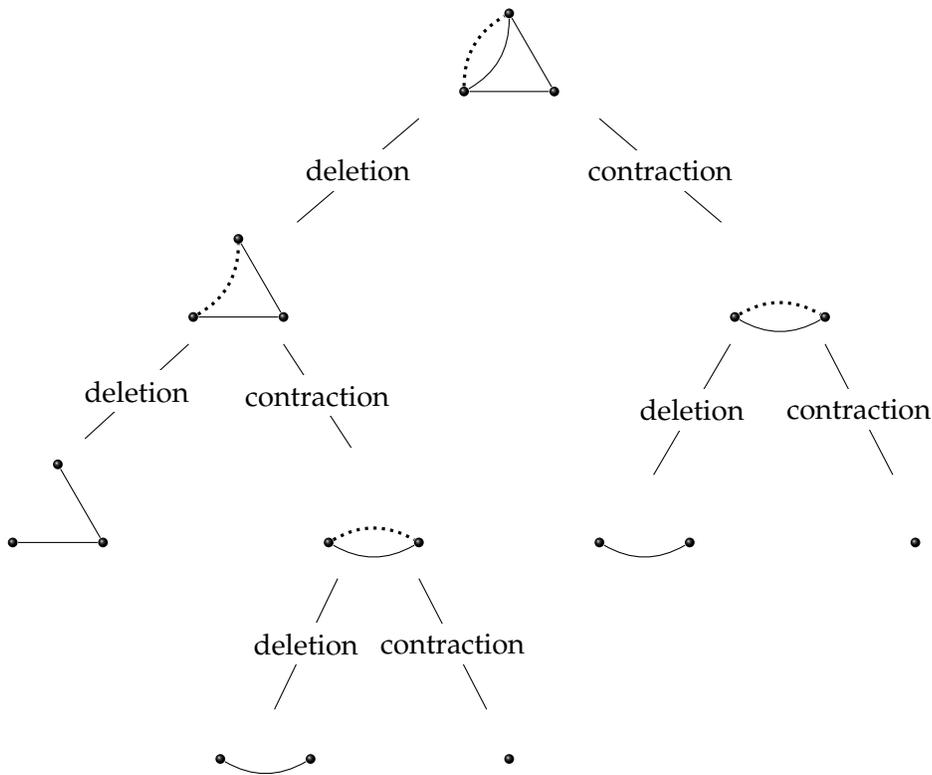
*Problem 2* (deletion and contraction). Let  $G$  be a multigraph, and let  $e$  be an edge of  $G$ . Define  $G - e$  to be the graph obtained from  $G$  by removing the edge  $e$  (but retaining the endpoints of  $e$ ). Let  $G/e$  be the graph obtained from  $G$  by “contracting” the edge  $e$ . To contract  $e$ , remove  $e$  from  $G$  and then glue the endpoints of  $e$  together to make a single vertex from the two vertices. If there were multiple edges between the endpoints of  $e$ , loops will be formed, but for our purposes, we remove these loops as in the following:



- (a) For an arbitrary connected multigraph  $G$ , choose an edge  $e$  such that  $G - e$  is connected. Let  $T(G)$ ,  $T(G - e)$ , and  $T(G/e)$  denote the number of spanning trees of  $G$ ,  $G - e$ , and  $G/e$ , respectively. The spanning trees of  $G$  come in two types: those that contain  $e$  and those that do not. Use that idea to prove

$$T(G) = T(G - e) + T(G/e).$$

- (b) We can use the previous problem iteratively to count spanning trees. This is illustrated in the diagram below (at each stage, the edge chosen to delete and contract is dotted):



We stop in this deletion-contraction process when there are no edges left whose removal would leave a connected graph. Along the bottom, there are 5 trees (a single isolated vertex is considered to be a tree, too). The previous part of this problem implies there are 5 spanning trees of the original graph. These are the 5 spanning trees we saw earlier.

Make a similar diagram for the graph in Problem 1. (This diagram should verify the number of spanning trees you found earlier.)<sup>1</sup>

<sup>1</sup>The first part of Problem 2 implies the amazing fact that number of trees at the bottom of the diagram is independent of the choices of edges made in constructing the diagram!