PROBLEM 1. Recall that a forest is an acyclic graph, *i.e.*, a graph which contains no cycle as a subgraph. Suppose G is a forest with 50 vertices and 44 edges. How many connected components does the G have?

SOLUTION: Let k be the number of connected components, and let $n_1, \ldots, n_k$ be the number of vertices in each of them. Since each connected component is a tree, we know that the ith connected component has $n_i - 1$ edges. We then have

$$50 = \sum_{i=1}^{k} n_i,$$

but also

$$44 = \sum_{i=1}^{k} (n_i - 1) = \left(\sum_{i=1}^{k} n_i \right) - k = 50 - k.$$

Solving for k we get $k = 6$.

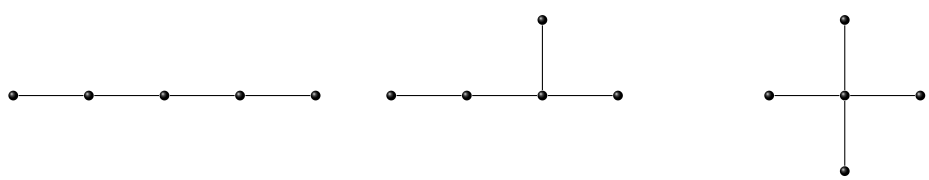
PROBLEM 2.

(a) Find the three unlabeled trees with five vertices.

(b) Use these unlabeled trees to count the number of (labeled) trees with five vertices.

SOLUTION:

(a) The three unlabeled trees are as follows.

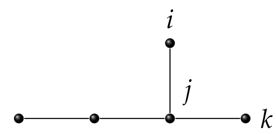


T_1 T_2 T_3

(b) We can label the vertices of T_1 with $1, 2, 3, 4, 5$ in $5! = 120$ ways left-to-right, but each labeled tree that arises this way is counted twice. For instance the following two trees are equal (not just isomorphic—they have the same set of vertices and the same set of edges):

Therefore, there are $120/2 = 60$ labeled trees arising from T_1.

Next, consider T_2.



T_2

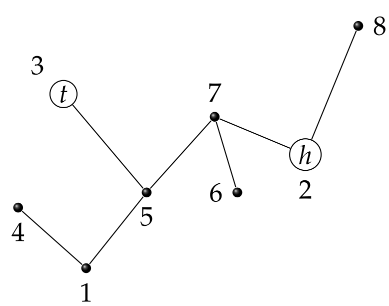
There are 5 ways to choose a label for vertex j. We then need to choose a pair of labels for i and k. The order in which we assign these labels does not matter: if we swap the labels on i and k, we get graphs that are equal. Therefore, there are $\binom{4}{2} = 6$ ways to choose i and k from the remaining 4 labels (as opposed to $4 \cdot 3 = 12$). That leaves 2 labels to assign, and order matters. So the number of trees arising from T_2 is $5 \cdot 6 \cdot 2 = 60$.

Finally, a labeling of T_3 is determined by choosing a label for its central vertex. Once we have done that, no matter how we label the remaining 4 vertices, we get the same labeled graph (check: we will get the same vertices and same edges). This gives 5 more labeled trees.

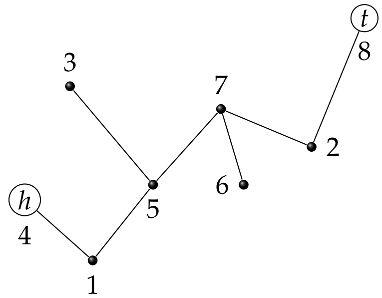
The total is $60 + 60 + 5 = 125 = 5^3 = 5^{5-2}$, in accordance with Cayley's formula.

PROBLEM 3. Determine the functions $[8] \to [8]$ associated with the following vertebrates:

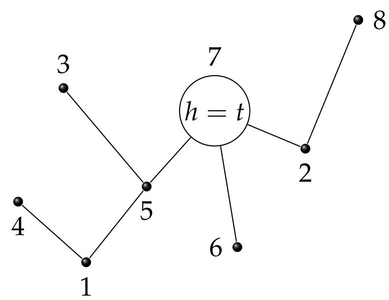
(a)



(b)



(c)



SOLUTION:

(a) First define the value of the function along the spine:

i	2	3	5	7
$f(i)$	3	5	7	2

Next, direct the edges of the appendages towards the spine, and read off the rest of the function:

i	1	2	3	4	5	6	7	8
$f(i)$	5	3	5	1	7	7	2	2

(b) First define the value of the function along the spine:

i	1	2	4	5	7	8
$f(i)$	8	2	7	5	1	4

Next, direct the edges of the appendages towards the spine, and read off the rest of the function:

i	1	2	3	4	5	6	7	8
$f(i)$	8	2	5	7	5	7	1	4

(c) First define the value of the function along the spine:

i	7
$f(i)$	7

.

Next, direct the edges of the appendages towards the spine, and read off the rest of the function:

i	1	2	3	4	5	6	7	8
$f(i)$	5	7	5	1	7	7	7	2

.

PROBLEM 4. Find the vertebrates associated with the following functions

(a)

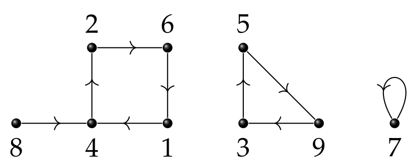
i	1	2	3	4	5	6	7	8	9
$f(i)$	4	6	5	2	9	1	7	4	3

(b)

i	1	2	3	4	5	6	7	8	9
$f(i)$	2	3	1	5	6	1	8	8	8

SOLUTION:

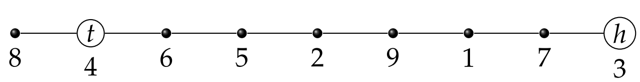
(a) The directed graph associated with f:



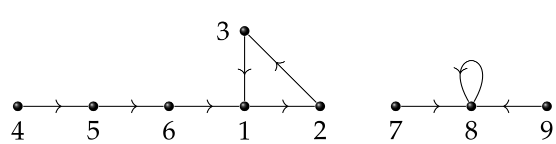
Every vertex except 8 is in a cycle. So these are the cycle vertices. Restricting the function to these vertices, we get

i	1	2	3	4	5	6	7	9
$f(i)$	4	6	5	2	9	1	7	3

The bottom row of this table gives the spine, and then $f(8) = 4$ tells us how to attach 8 to the vertebrate:



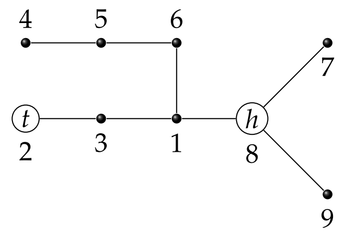
(b) The directed graph associated with f:



The vertices in cycles are $1, 2, 3, 8$. Restricting the function to these vertices, we get

i	1	2	3	8
$f(i)$	2	3	1	8

The bottom row of this table gives the spine, and we then for each vertex i not in a cycle, we attach the edge $\{i, f(i)\}$.



PROBLEM 5. Characterize the vertebrates associated with functions $[n] \to [n]$ which are permutations (i.e., bijective).

SOLUTION: In the directed graph associated to a permutation f, every vertex is part of a cycle. (This requires some thought. Imagine drawing the directed graph. If there are vertices that are not part of a cycle, then there is at least one vertex i with only one edge directed out of it and no edges directed into it. This would mean that i is not in the image of f, even though f is bijective. That cannot happen.) Thus, every vertex is on the spine. Therefore, the corresponding vertebrate is a path graph: