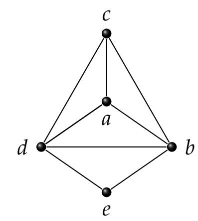
PROBLEM 1. Consider the graph pictured in the margin.



(a) Find a path of maximal length. (Recall: a path contains no repeated vertices.)

(b) Find a cycle containing all of the vertices.

(c) Find an Eulerian walk from a to c.

(d) Find a Hamiltonian cycle.

SOLUTION:

(a) By the pigeonhole principle, since we have only five vertices, a maximal length path has at most four edges (and thus five vertices). One example of a maximal-length path is *a-b-e-d-c*.

(b) One such is *a-b-e-d-c-a*.

(c) The graph has exactly two vertices of odd degree: a and c. Thus, it has no closed Eulerian walk, but it does have an Eulerian walk. Any such walk must start at one of these two vertices and end at the other. An example is *a-b-e-d-b-c-d-a-c*.

(d) *a-b-e-d-c-a*.

PROBLEM 2. Let G be a graph.

(a) Let u and v be two vertices of G. Prove that if there is a walk in G from u to v, then there is a path in G from u to v.

(b) Define a relation on the set V of vertices of G as follows: $u \sim v$ if there exists a path in G from u to v. Prove that $\sim$ is an equivalence relation on V.

SOLUTION:

(a) Let $v_0 v_1 \cdots v_\ell$ be the walk from $u = v_0$ to $v = v_\ell$. If this walk is already a path, we are done. If not, there exist $i < j$ such that $v_i = v_j$. Note that then $v_0 \cdots v_i v_{j+1} \ldots v_n$ is still a walk from u to v (if $\ell = j$, we have that $v_0 \cdots v_i$ is a walk from u to $v = v_\ell = v_i$). We can repeat this process until there are no repeated vertices, thus getting a path.

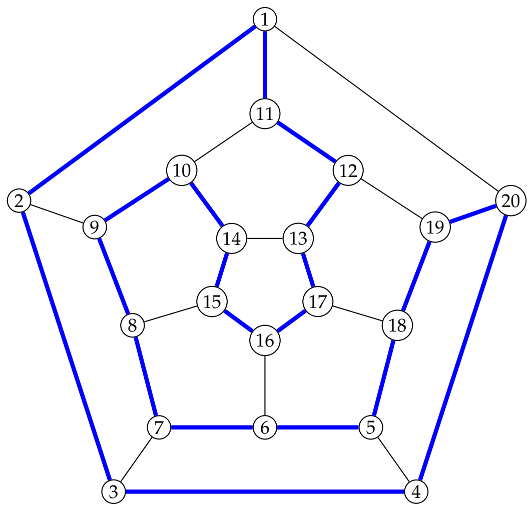
(b) *Reflexive:* Let $u \in V$. Then u is a path from u to itself, hence $u \sim u$.

Symmetric: Suppose $u \sim v$. Thus there exists a path $v_0 \cdots v_\ell$ from $u = v_0$ to $v = v_\ell$. Note that the sequence $v_\ell \cdots v_0$ (where we reverse the order) is also a path since consecutive vertices are still connected by an edge. This is a path from $v = v_\ell$ to $u = v_0$.

Transitive: Suppose $u \sim v$ and $v \sim w$. Thus there exist paths $v_0 \cdots v_\ell$ and $w_0 \cdots w_m$ with $v_0 = u$, $v_\ell = v = w_0$ and $w_m = w$. Note then that $v_0 \ldots v_\ell w_1 \ldots w_m$ is a *walk** from $u = v_0$ to $w = w_m$. By part (i), there exists a path from u to w, so $u \sim v$.

* There might be repeated vertices between the two paths we are concatenating), so we might not get a path.

PROBLEM 3. Does the dodecahedron graph have a Hamiltonian cycle? If so, demonstrate one by listing its vertices.



SOLUTION: One example is highlighted above. Are there others that do not come from this one by symmetry?