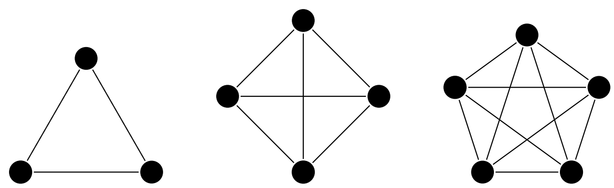
PROBLEM 1. A *complete graph* on n vertices, denoted K_n, has every possible edge. Draw pictures of K_3, K_4, and K_5. How many edges are there in a complete graph on n vertices? What it the maximal number of edges for a graph G with vertex set V? What is the minimal number of edges for a graph G with vertex set V?

SOLUTION: Here are pictures of K_n, $n = 1, 2, 3$.



There are $\binom{n}{2} = \frac{n(n-1)}{2}$ edges in K_n. Since this is the maximal number of edges, we know that

$$|E| \leq \frac{|V|(|V|-1)}{2}$$

in any graph $G = (V, E)$, with equality being achieved in the case of the complete graph.

The minimal number of edges of a graph with V vertices is 0. A graph is not required to have any edges.

PROBLEM 2. A graph $G = (V, E)$ is called *bipartite* if it is possible to partition V with nonempty sets as $V = A \amalg B$ such that edges only go between A and B. The *complete bipartite graph on $p + q$ vertices*, denoted $K_{p,q}$, has $|A| = p$, $|B| = q$, and all possible edges between A and B.

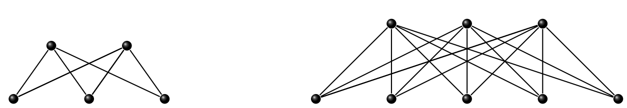
(a) Draw pictures of $K_{2,3}$ and $K_{3,5}$.

(b) How many edges are in $K_{p,q}$?

(c) If $|A| = p$ and $|B| = q$ with $A \cap B = \varnothing$, how many (not necessarily complete) bipartite graphs have vertex set $A \cup B$ with $A \amalg B$ as the specified partition?

SOLUTION:

(a) Here are the pictures of $K_{2,3}$ and $K_{3,5}$.



(b) Each of the p vertices in A must be connected to each of the q vertices in B, so by the multiplicative counting principle there are pq edges.

Alternatively, there are p vertices with degree q and q vertices with degree p. Thus the total degree is $pq + qp = 2pq = 2|E|$ (by the handshake theorem). We conclude that $K_{p,q}$ has pq edges.

(c) Such a graph is given by selecting a subset of the edges in $K_{p,q}$, so there are

$$2^{pq}$$

of them.

PROBLEM 3. The definition of graph isomorphism implies that isomorphic graphs have the same number of vertices and same number of edges.

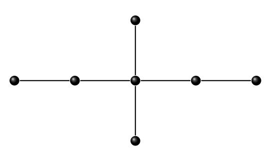
(a) Must two graphs with the same number of vertices and same number of edges be isomorphic? Prove it or find a counterexample?

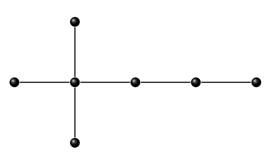
(b) The *degree sequence* of a graph is a list of its vertex degrees in non-decreasing order. Prove that graphs with the same degree sequence have the same number of edges.

(c) Must two graphs with the same degree sequences be isomorphic? Prove it or find a counterexample.

SOLUTION:

(a) They need not be be isomorphic. The following two graphs both have seven vertices and six edges.





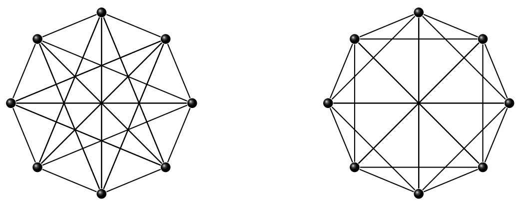
There are not isomorphic because, for instance, there are length 3 paths from leaf to leaf in the first graph, but no such paths in the second graph. (We leave it to the reader to check that such properties are invariant under isomorphism.)

(b) This follows from the handshake theorem. The total degrees are the same, and thus the numbers of edges are the same.

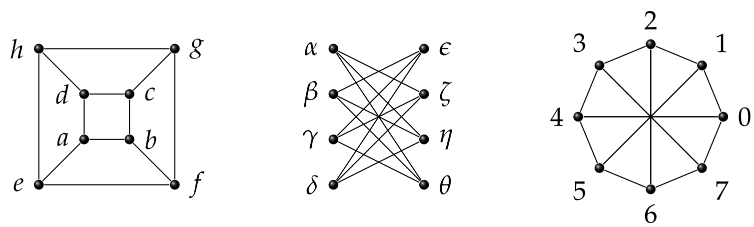
(c) They need not be isomorphic. The graphs pictured in (a) provide an example as they both have degree sequence $(1, 1, 1, 2, 2, 4)$.

PROBLEM 4.

(a) Determine whether the following graphs are isomorphic.

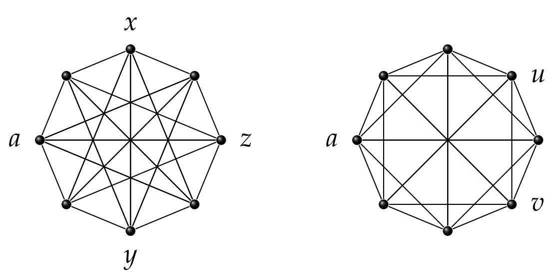


(b) Determine whether the graphs in any pair of the following are isomorphic.



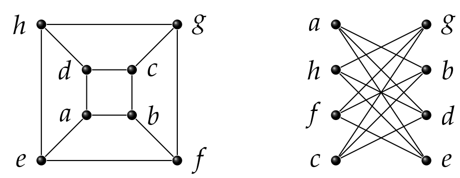
SOLUTION:

(a) These graphs are not isomorphic. Consider the labelings below:



By symmetry of both graphs, we may assume that any possible isomorphism would send the vertex a on the left graph to vertex a on the right graph. The points x and y are the only points not connected to a on the left, and similarly for points u and v on the right. Thus, any isomophism would have to send x and y to u and v in some order. But now note that the point z is not connected by an edge to either x or y whereas *every* point on the right besides except a is adjacent to either u or v on the right.

(b) The first two graphs in (b) are isomorphic, as shown by the following relabeling of the second graph:



One may easily check that with this relabeling, the sets of edges are identical.

The third graph is not isomorphic to the first two. If it were, it would be bipartite with vertices partitioned into two parts A and B. Say $0 \in A$. Since $1, 4, 7$ are adjacent to 0, this means that $1, 4, 7 \in B$. Since $3, 5$ are adjacent to 4, we have $3, 5 \in A$. Since 5 is adjacent to 6, it must be that $6 \in B$. But then $6, 7 \in B$ and they are adjacent.