See back for hints (but only after trying the problems for a while).

PROBLEM 1. In a round robin chess tournament with n participants, every player plays every other player exactly once. Prove that at any given time during the tournament, two players have finished the same number of games.

PROBLEM 2. What is the least number of area codes needed to guarantee that the 25 million phones in a state can be given distinct 10-digit telephone numbers of the form NXX-NXX-$XXXX$ where each X is any digit from 0 to 9 and each N represents a digit from 2 to 9? (The area code is the first three digits.)

PROBLEM 3. Show that in the sequence $7, 77, 777, 7777, \ldots$ there is an integer divisible by 2003.

Hints for problem 1:

(a) What is the minimum m and maximum M number of games that a player has played at any point in the tournament? (You will see that our problem looks like a pigeon-hole problem in which there are n pigeons and n boxes, but read on for something clever.)

(b) At any point in the tournament, either there exists a player who has played M games, or there is no player who has played M games.

   (i) Suppose that at some point, a player has played M games. What is the minimum and maximum number of games that the other players have played at that point.

   (ii) What if at some point no player has played M games? What is the minimum and maximum number of games that any of the players has played?

Hints for problem 3:

(a) Let a_i and a_j be in the sequence with $a_i > a_j$. Show that $a_i - a_j = a_k 10^r$ for some natural number r. Use this fact to show that if 2003 divides $a_i - a_j$, then it divides a_k.

(b) How many possible remainders does a_i have upon division by 2003?