PROBLEM 1. In a round robin chess tournament with $n$ participants, every player plays every other player exactly once. Prove that at any given time during the tournament, two players have finished the same number of games.

Hints:

(a) What is the minimum $m$ and maximum $M$ number of games that a player has played at any point in the tournament? (You will see that our problem looks like a pigeon-hole problem in which there are $n$ pigeons and $n$ boxes, but read on for something clever.)

(b) At any point in the tournament, either there exists a player who has played $M$ games, or there is no player who has played $M$ games.

   (i) Suppose that at some point, a player has played $M$ games. What is the minimum and maximum number of games that the other players have played at that point.

   (ii) What if at some point no player has played $M$ games? What is the minimum and maximum number of games that any of the players has played?

SOLUTION: At any given moment, each player has played between 0 and $n-1$ games, a range of $n$ possibilities, so the pigeonhole principle does not directly apply. Note, though, that if one player has played $n-1$ games, then everyone has played between 1 and $n-1$ games, a range of $n-1$ possibilities. If no players have played $n-1$ games, then everyone has played between 0 and $n-2$ games, again $n-1$ possibilities. Thus the pigeonhole principle applies in both cases to guarantee that (at least) two players have played the same number of games.

PROBLEM 2. What is the least number of area codes needed to guarantee that the 25 million phones in a state can be given distinct 10-digit telephone numbers of the form $NXX$-$NXX$-$XXXX$ where each $X$ is any digit from 0 to 9 and each $N$ represents a digit from 2 to 9? (The area code is the first three digits.)

SOLUTION: There are $8 \cdot 10^6$ seven-digit phone numbers (excluding area code) according to these rules. With 3 or fewer area codes, there are at most 24 million distinct phone numbers, whence the pigeonhole principle would guarantee phone number repetition in the state. With 4 area codes, there are 32 million distinct phone numbers, a sufficient number to prevent repetition.

PROBLEM 3. Show that in the sequence $7, 77, 777, 7777, \ldots$ there is an integer divisible by 2003.

Hints:

(a) Let $a_i$ and $a_j$ be in the sequence with $a_i > a_j$. Show that $a_i - a_j = a_k 10^r$ for some natural number $r$. Use this fact to show that if 2003 divides $a_i - a_j$, then it divides $a_k$.

(b) How many possible remainders does $a_i$ have upon division by 2003?

SOLUTION: Following the hint, suppose $a_i > a_j$ are terms of the sequence such that $a_i - a_j$ is divisible by 2003. The number $a_i - a_j$ is of the form $a_k \cdot 10^r$ for some positive integer $r$. Since 2003 does not share any prime factors with 10 (in fact, 2003 is prime), we have that 2003 divides $a_k$.

Now note that when we divide a term $a_i$ by 2003, we get a remainder between 0 and 2002. If the remainders of terms $a_i$ and $a_j$ are equal, then $a_i - a_j = 2003q_i + r - (2003q_j + r) = 2003(q_i - q_j)$ for some integers $q_i, q_j, r$. Thus 2003 divides $a_i - a_j$. Finally, note that there are finitely many remainders and infinitely many terms $a_i > a_j$, so such a pair with common remainder must exist.