PROBLEM 1. A large software development company employs 100 computer programmers. Of them, 45 are proficient in Java, 30 in C++, 20 in Python, six in C++ and Java, one in Java and Python, five in C++ and Python, and just one programmer is proficient in all three languages above. Determine the number of computer programmers that are not proficient in any of these three languages.

SOLUTION: Let E denote the set of employees, and let J, C and P denote the sets of employees proficient in Java, C+ and Python, respectively. We are interested in knowing $|E \setminus (J \cup C \cup P)|$. We know:

$$\begin{array}{cccc} |E| = 100 & |J| = 45 & |C| = 30 & |P| = 20 \\ |J \cap C| = 6 & |J \cap P| = 1 & |C \cap P| = 5 & |J \cap C \cap P| = 1. \end{array}$$

Using the principle of inclusion-exclusion, we have

$$\begin{aligned} |J \cup C \cup P| &= |J| + |C| + |P| - |J \cap C| - |J \cap P| - |C \cap P| + |J \cap C \cap P| \\ &= 45 + 30 + 20 - 6 - 1 - 5 + 1 \\ &= 84. \end{aligned}$$

Thus, there are 84 employees that are proficient in at least one of these languages, so there are 100-84=16 employees who are not proficient in any of these three.

PROBLEM 2. How many poker hands (5 cards) from a regular deck (52 cards) have at least one card from each of the four standard suits? *Hint*: Let $N_\spadesuit$ be the collection of hands containing no spades, and similarly define $N_\clubsuit$, $N_\heartsuit$, and $N_\diamondsuit$. What is the relationship between the answer to this question and $|N_\spadesuit \cup N_\clubsuit \cup N_\heartsuit \cup N_\diamondsuit|$?

SOLUTION: Let S denote the set of hands with at least one card from each suit, and let H denote the set of all hands. Then

$$S = H \smallsetminus (N_\spadesuit \cup N_\clubsuit \cup N_\heartsuit \cup N_\diamondsuit)$$

and

$$|S| = |H| - |N_\spadesuit \cup N_\clubsuit \cup N_\heartsuit \cup N_\diamondsuit|.$$

Since each hand contains 5 of the 52 cards, $|H| = \binom{52}{5}$, and it remains to count $|N_\spadesuit \cup N_\clubsuit \cup N_\heartsuit \cup N_\diamondsuit|$.

We proceed via inclusion-exclusion. Since only the excluded suit changes, we have $|N_\spadesuit| = |N_\clubsuit| = |N_\heartsuit| = |N_\diamondsuit|$, and for each of these counts we select 5 cards from the $52 - 13 = 39$ cards which aren't of the selected suit. Thus the cardinality of each of these is $\binom{39}{5}$. Each pairwise intersection excludes 26 cards and thus has cardinality $\binom{26}{5}$, and each triple intersection excludes 39 cards and thus has cardinality $\binom{13}{5}$. The quadruple intersection is empty, since each card has some suit. Note that there are $\binom{4}{2} = 6$ pairwise intersections and there are $\binom{4}{3} = 4$ triple intersections. We conclude that

$$|N_{\spadesuit} \cup N_{\clubsuit} \cup N_{\heartsuit} \cup N_{\diamondsuit}| = 4 \cdot \binom{39}{5} - 6 \cdot \binom{26}{5} + 4 \cdot \binom{13}{5}$$

and

$$|S| = \binom{52}{5} - 4 \cdot \binom{39}{5} + 6 \cdot \binom{26}{5} - 4 \cdot \binom{13}{5} = 685,464.$$

We can also proceed without using the inclusion-exclusion principle. Every such hand can be constructed by choosing a spade, then a club, then a heart, then a diamond, and then one of the remaining 48 cards. This results in $13^4 \cdot 48$ choices, but overcounts in that the final card may be swapped with the other card of its suit, resulting in the same hand. (Hands don't have an order.) Thus there are

$$\frac{13^4 \cdot 48}{2} = 685,464$$

such hands.

Here's one more way to approach the problem: In order to construct such a hand, we first choose any of the 52 cards and note its suit. We then choose any of the remaining 39 cards of a different suit, then any of the remaining 26 cards not of the first two suits, then any of the remaining 13 cards not of the first 3 suits. Finally, we choose any of the remaining 48 cards. All such hands can be produced in this way, but there are still 4! to permute the first four cards and 2 ways to swap (or not swap) the final card with the one matching its suit. Thus there are

$$\frac{52 \cdot 39 \cdot 26 \cdot 13 \cdot 48}{4! \cdot 2} = 685,464$$

such hands.

PROBLEM 3. Let m and n be integers greater or equal to 1. How many surjective functions $f\colon [m] \to [n]$ are there?

SOLUTION: First note that if $m < n$ there are no surjective functions $f\colon [m] \to [n]$ because the image of a function from $[m]$ has at most m elements. Thus let $m \geq n$. Let F denote the set of all functions and S denote the set of surjective functions. Note that $F \setminus S$ is the set of non-surjective functions, i.e.,

$$F \setminus S = \{f\colon [m] \to [n] \mid \text{there exists } i \in [n] \setminus \operatorname{im}(f)\}.$$

For each $i \in [n]$, let

$$A_i = \{f\colon [m] \to [n] \mid i \notin \operatorname{im} f\}.$$

Thus,

$$F \setminus S = A_1 \cup A_2 \cup \cdots \cup A_n.$$

We calculate the cardinality of this set via inclusion-exclusion. Note that all the A_i's will have the same cardinality, namely $(n-1)^m$, since we are looking at functions that miss i, so the set of possible values of the function has $n-1$ elements.

In general, a k-tuple intersection of A_i's will have cardinality $(n-k)^m$. To explain this, with out loss of generality take $A_1 \cap \cdots \cap A_k$. This is the set of functions that miss 1 AND 2 AND ... AND k, so we are looking at functions that land in $\{k+1, \ldots, n\}$. There are $n-k$ options for each of the m elements in the domain $[m]$, and hence there are $(n-k)^m$ of these functions. Now also recall that there are $\binom{n}{k}$ of these intersections.

Using inclusion-exlusion we get:

$$\begin{aligned}
|A_1 \cup A_2 \cup \cdots \cup A_n| &= \binom{n}{1}(n-1)^m - \binom{n}{2}(n-2)^m + \ldots \\
&\quad + (-1)^{k-1}\binom{n}{k}(n-k)^m + \ldots \\
&\quad + (-1)^{n-1}\binom{n}{n}(n-n)^m \\
&= \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}(n-k)^m.
\end{aligned}$$

Finally, recall we are really interested in the cardinality of S, so

$$\begin{aligned}
|S| &= |F| - |A_1 \cup A_2 \cup \cdots \cup A_n| \\
&= n^m - \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}(n-k)^m \\
&= n^m + \sum_{k=1}^{n}(-1)^{k}\binom{n}{k}(n-k)^m \\
&= \sum_{k=0}^{n}(-1)^{k}\binom{n}{k}(n-k)^m.
\end{aligned}$$

The last step absorbs the term n^m into the sum with the index $k = 0$.