PROBLEM 1. Use induction to show that

$$2^0 + 2^1 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$$

for $n \geq 1$. Write a complete proof using the template from our text as a guide.

PROBLEM 2. Let $n \geq 1$. Consider a $2^n \times 2^n$ chessboard and remove one of the corner squares. Prove that the remaining board can be tiled with L-shaped trominoes. (You might want to start by tiling a 4×4 board, and then an 8×8 board.)

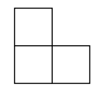

Figure 1: An L-shaped tromino.

PROBLEM 3. Let $m \geq 1$ and $1 \leq r \leq m$. Let $s(m, r)$ denote the number of ways to write m as a sum of r positive numbers. We will use induction to prove $s(m, r) = \binom{m-1}{r-1}$. *

(a) Prove that for all m, $s(m, 1) = 1$ and $s(m, m) = 1$.

(b) Assume that $m \geq 1$ and $2 \leq r \leq m$. Prove that

$$s(m+1, r) = s(m, r-1) + s(m, r).$$

(*Hint:* Given a way of writing $m + 1$ as a sum, either the first term is 1 or it's not. Try to count both cases separately.)

(c) Conclude using induction.

* You might find the following identity helpful:

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

PROBLEM 4. Use induction to prove that the number of diagonals in a convex n-gon is $n(n-3)/2$ for $n \geq 3$.

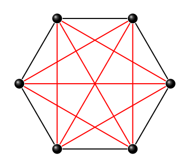

Figure 2: A hexagon has $\frac{6(6-3)}{2} = 9$ diagonals.

Challenge

Challenge problems are optional and should only be attempted after completing the previous problems.

Using induction, we can prove that in every gathering of Reed students, all the students have the same hair color. The formal statement is: If X is a set of n Reed students, then all the students in X have the same hair color.

We induct on the size of the set of students in the gathering. The base case of $n = 1$ is clear. So assume the result holds for some $n \geq 1$. Let X be a set of Reed students of size $n + 1$. Choose a student $A \in X$. Removing that student from X produces the set $X \setminus \{A\}$ of size n. By induction, all of these students have the same hair color H_1. Now remove a different student B from X. By induction, again, all the students in $X \setminus \{B\}$ have the same hair color H_2. Notice that $A \in X \setminus B$, and therefore has hair color H_2. Similarly, B has hair color H_1. Now for the interesting part: Let $C \in X$ be a student who has not been chosen, yet, i.e., C is neither A nor B. Since $C \in X \setminus A$, we know C's hair color is H_1. Similarly, since $C \in X \setminus B$, we know C's hair color is H_2. It follows that $H_1 = H_2$. We have accounted for every student in X and shown they have the same hair color. The result now follows by induction. What, precisely, is wrong with this argument?