PROBLEM 1. Use induction to show that

$$2^0 + 2^1 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$$

for $n \geq 1$. Write a complete proof using the template from our text as a guide.

SOLUTION: We will prove this by induction. The statement holds for the base case, $n = 1$, since

$$2^0 = 2^1 - 1.$$

Next, suppose the statement holds for some $n \geq 1$. It follows that

$$\begin{aligned} 2^0 + 2^1 + 2^2 + \cdots + 2^n &= (2^0 + 2^1 + 2^2 + \cdots + 2^{n-1}) + 2^n \\ &= 2^n + 2^n = 2 \cdot 2^n \\ &= 2^{n+1}, \end{aligned}$$

and the result holds for $n + 1$, too. Hence, the statement holds for all $n \geq 1$ by induction.

PROBLEM 2. Let $n \geq 1$. Consider a $2^n \times 2^n$ chessboard and remove one of the corner squares. Prove that the remaining board can be tiled with L-shaped trominoes. (You might want to start by tiling an 8×8 board.)

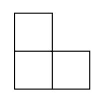
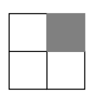


Figure 1: An L-shaped tromino.

SOLUTION: We will prove the statement by induction. Let $n = 1$. In that case we have a 2×2 board with one corner removed, which can clearly be tiled with exactly one L-shaped tromino:



Now assume that for some $n \geq 1$ we can indeed tile the $2^n \times 2^n$ chessboard with one corner removed. Consider the $2^{n+1} \times 2^{n+1}$ chessboard with one corner removed, and split it into four $2^n \times 2^n$ chessboards, one of which has a corner removed. For the other three, remove the corners that meet at the center of the large chessboard. By the inductive hypothesis, each of the smaller chessboards can be tiled with L-shaped trominoes. The three corners that were removed form an L-shape, and thus can be covered with a final tile, proving that we can tile the $2^{n+1} \times 2^{n+1}$ chessboard with one corner removed. This completes the proof by induction.

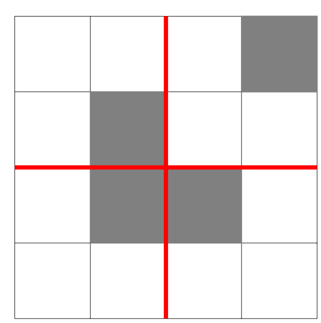


Figure 2: The inductive step from n=1 to n=22.

PROBLEM 3. Let $m \geq 1$ and $1 \leq r \leq m$. Let $s(m,r)$ denote the number of ways to write m as a sum of r positive numbers. We will use induction to prove $s(m,r) = \binom{m-1}{r-1}$. *

(a) Prove that for all m, $s(m,1) = 1$ and $s(m,m) = 1$.

* You might find the following identity helpful:

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

(b) Assume that $m \geq 1$ and $2 \leq r \leq m$. Prove that

$$s(m+1, r) = s(m, r-1) + s(m, r).$$

(*Hint:* Given a way of writing $m+1$ as a sum, either the first term is 1 or it's not. Try to count both cases separately.)

(c) Conclude using induction.

SOLUTION:

(a) When $r = 1$, there is only one way to write m as a sum of 1 term, namely $m = 1$. When $r = m$ there is also only one way: as a sum of m 1's. Therefore $s(m, 1) = s(m, m) = 1$.

(b) We will partition the number of ways in which we can write $m+1$ as a sum according to the first term. Note that if the first term is 1, the remaining numbers give a way to write m with $r-1$ terms, thus, there are $s(m, r-1)$ of those. If the first term is $k > 1$, we can replace k with $k-1$. That gives a way to write m with r terms, so there are $s(m, r)$ of those. By the additive counting principle, it follows that

$$s(m+1, r) = s(m, r-1) + s(m, r).$$

(c) We will proceed by induction on m. When $m = 1$, we must have $r = 1$, and by part (i) we have

$$s(1, 1) = 1 = \binom{0}{0},$$

as wanted. Now assume the statement holds for some $m \geq 1$. Part (i) gives us the result we want if $r = 1$ or $r = m+1$:

$$s(m+1, 1) = 1 = \binom{m}{0} \quad \text{and} \quad s(m+1, m+1) = 1 = \binom{m}{m}.$$

Now, for $2 \leq r \leq m$, part (ii) together with the inductive hypothesis and the identity in the footnote give:

$$\begin{aligned} s(m+1, r) &= s(m, r-1) + s(m, r) \\ &= \binom{m-1}{r-2} + \binom{m-1}{r-1} \\ &= \binom{m}{r-1}. \end{aligned}$$

Thus the result holds for $m+1$ as well. The statement follows for all $m \geq 1$ by induction.

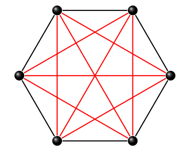


Figure 3: A hexagon has $\frac{6(6-3)}{2} = 9$ diagonals.

PROBLEM 4. Use induction to prove that the number of diagonals in a convex n-gon is $n(n-3)/2$.

SOLUTION: Our base case is $n = 3$, the triangle, which has no diagonals, and indeed $3(3-3)/2 = 0$. Fix $n \geq 3$ and suppose for induction that a convex n gon has $n(n-3)/2$ diagonals. Now consider a convex $(n+1)$-gon with vertices labeled $1, 2, \ldots, n+1$ in order. By the inductive hypothesis, the n-gon with vertices $1, \ldots, n$ has $n(n-3)/2$ diagonals, and each of these is a diagonal of our $(n+1)$-gon. Additionally, the $(n+1)$-gon has diagonals joining $n+1$ to $2, 3, \ldots, n-1$, and it also has the diagonal from 1 to n. That amounts to $n-1$ additional diagonals, so the $(n+1)$-gon has

$$\frac{n(n-3)}{2} + n - 1 = \frac{(n^2 - 3n) + (2n-2)}{2} = \frac{(n+1)((n+1)-3)}{2}$$

diagonals, as desired.

Challenge

Challenge problems are optional and should only be attempted after completing the previous problems.

Using induction, we can prove that in every gathering of Reed students, all the students have the same hair color. The formal statement is: If X is a set of n Reed students, then all the students in X have the same hair color.

We induct on the size of the set of students in the gathering. The base case of $n = 1$ is clear. So assume the result holds for some $n \geq 1$. Let X be a set of Reed students of size $n+1$. Choose a student $A \in X$. Removing that student from X produces the set $X \setminus \{A\}$ of size n. By induction, all of these students have the same hair color H_1. Now remove a different student B from X. By induction, again, all the students in $X \setminus \{B\}$ have the same hair color H_2. Notice that $A \in X \setminus B$, and therefore has hair color H_2. Similarly, B has hair color H_1. Now for the interesting part: Let $C \in X$ be a student who has not been chosen, yet, i.e., C is neither A nor B. Since $C \in X \setminus A$, we know C's hair color is H_1. Similarly, since $C \in X \setminus B$, we know C's hair color is H_2. It follows that $H_1 = H_2$. We have accounted for every student in X and shown they have the same hair color. The result now follows by induction. What, precisely, is wrong with this argument?

SOLUTION: The only problem with this argument is that the induction step assumes that n is at least two. (In other words words, the only thing that keeps all Reed students from having the same hair color is the case $n = 2$!)