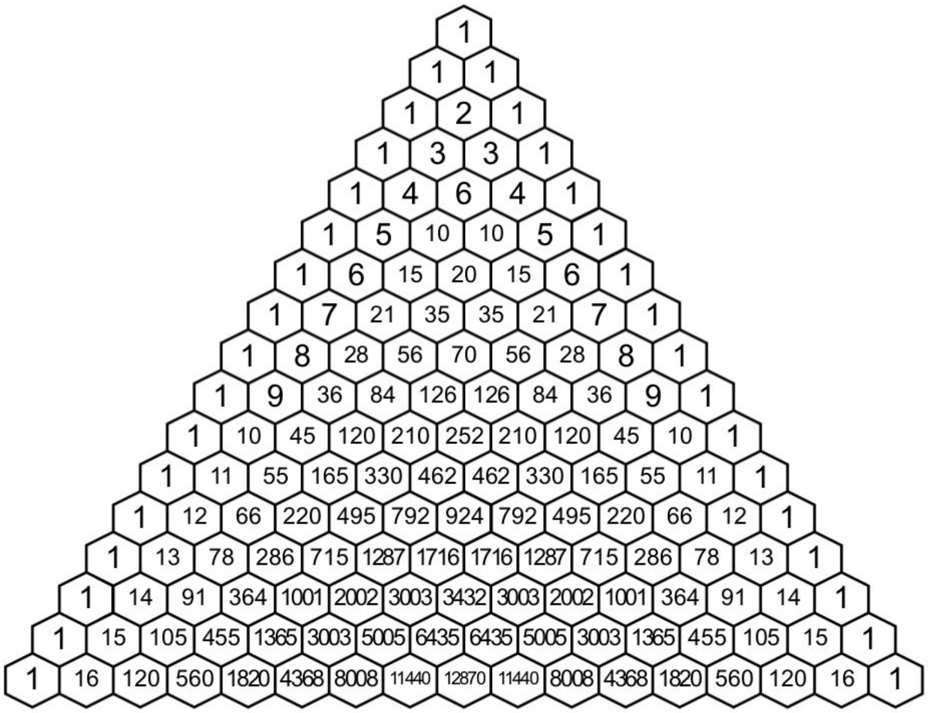
For reference, here is a copy of Pascal's triangle:



and here are two versions of the binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$(1+y)^n = \sum_{k=0}^{n} \binom{n}{k} y^k.$$

PROBLEM 1. The book claims that

$$\sum_{\ell=k}^{n} \binom{\ell}{k} = \binom{n+1}{k+1}$$

for all $k, n \in \mathbb{Z}$.

(a) Write out the above identity for the case $n = 5$ and $k = 2$.

(b) Highlight the terms involved in this identity for various k and n on Pascal's triangle; explain why it is known as the *hockey stick identity*. (Recall that the row's of Pascal's triangle are indexed starting with $n = 0$.)

(c) Let X be the set of subsets of $[n+1]$ of cardinality $k+1$, and let

$$X_a := \{A \in X \mid a \text{ is the first element of } [n+1] \text{ in } A\}$$

for $a = 1, 2, \ldots, n-k+1$. Check that the X_i partition X:

$$X = X_1 \amalg X_2 \amalg \cdots \amalg X_{n-k+1}.$$

(Is each $(k+1)$-subset of $[n+1]$ in exactly one X_i? We do we stop with the index $n-k+1$?)

(d) Determine the cardinality of X_a in terms of n, k, and a. Use this and (ii) to give a combinatorial proof of the hockey stick identity.

PROBLEM 2. In this problem we will answer the following question: how many ways are there to write a nonnegative integer m as a sum of r positive integer summands? (We decree that the order of the addends matters, so $3+1$ and $1+3$ are two different representations of 4 as a sum of 2 nonnegative integers.)

(a) Experiment with small cases: let $m = 1, 2, 3, 4$ and $1 \le r \le m$.

(b) Develop a conjecture.

(c) Prove your conjecture.

PROBLEM 3.

(a) Compute the sums

$$\begin{array}{r} \binom{0}{0}^2 \\ \binom{1}{0}^2+\binom{1}{1}^2 \\ \binom{2}{0}^2+\binom{2}{1}^2+\binom{2}{2}^2 \\ \binom{3}{0}^2+\binom{3}{1}^2+\binom{3}{2}^2+\binom{3}{3}^2 \\ \binom{4}{0}^2+\binom{4}{1}^2+\binom{4}{2}^2+\binom{4}{3}^2+\binom{4}{4}^2 \\ \binom{5}{0}^2+\binom{5}{1}^2+\binom{5}{2}^2+\binom{5}{3}^2+\binom{5}{4}^2+\binom{5}{5}^2 \end{array}$$

by hand and develop a conjecture regarding the value of

$$\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\cdots+\binom{n}{n-1}^2+\binom{n}{n}^2.$$

(b) Use the binomial theorem to prove your conjecture. [*Hint*: We have the identity $(1+y)^{2n} = (1+y)^n(1+y)^n$. Therefore, if we expand either side and find the coefficient of y^n, we will get the same number. Use the binomial theorem to find the coefficient of y^n in $(1+y)^{2n}$. Next apply the binomial theorem to $(1+y)^n$ and use the result to find the coefficient of y^n in $(1+y)^n(1+y)^n$.]

(c) Give a combinatorial argument proving your conjecture. [*Hint*: Split a set of size $2n$ into two pieces of size n, and then start building size n subsets of the original set.]

Challenge

Challenge problems are optional and should only be attempted after completing the previous problems.

Answer the variation of Problem 2 in which we allow *nonnegative* integer summands.