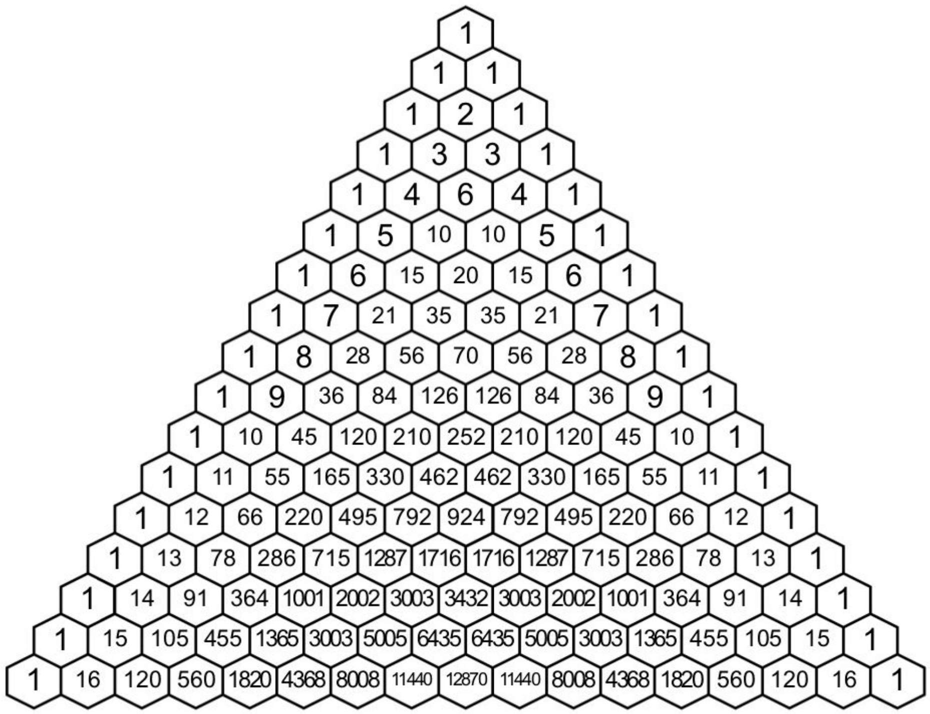
For reference, here is a copy of Pascal's triangle:



and here are two versions of the binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$(1+y)^n = \sum_{k=0}^{n} \binom{n}{k} y^k.$$

PROBLEM 1. The book claims that

$$\sum_{\ell=k}^{n} \binom{\ell}{k} = \binom{n+1}{k+1}$$

for all $k, n \in \mathbb{Z}$.

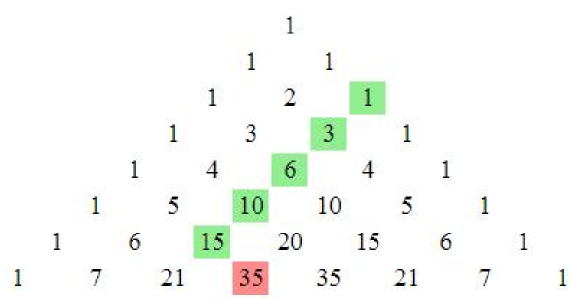
(a) Write out the above identity for the case $n = 5$ and $k = 2$.

SOLUTION: We have

$$\sum_{\ell=2}^{5} \binom{\ell}{2} = \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} = \binom{6}{3}.$$

(b) Highlight the terms involved in this identity for various k and n on Pascal's triangle; explain why it is known as the *hockey stick identity*.

SOLUTION: The hockey stick identity says that the terms in the "stick" add up to the "blade" term in the following picture:



(c) Let X be the set of subsets of $[n+1]$ of cardinality $k+1$, and let

$$X_a := \{A \in X \mid a \text{ is the first element of } [n+1] \text{ in } A\}$$

for $a = 1, 2, \ldots, n-k+1$. Check that

$$X = X_1 \amalg X_2 \amalg \cdots \amalg X_{n-k+1}.$$

(Is each $(k+1)$-subset of $[n+1]$ in exactly one X_i? We do we stop with the index $n-k+1$?)

SOLUTION: Clearly each X_a is a subset of X and they are disjoint since elements of X_a and X_b have different first elements when $a \neq b$. Additionally, every element of X has a first element between 1 and $n-k+1$ (no higher since the cardinality is $k+1$). This shows that $\{X_1, \ldots, X_{n-k+1}\}$ is a partition of X.

(d) Determine the cardinality of X_a in terms of n, k, and a. Use this and (ii) to give a combinatorial proof of the hockey stick identity.

SOLUTION: Elements of X_a are uniquely determined by choosing k elements from $\{a+1, a+2, \ldots, n+1\}$. We have

$$|\{a+1, a+2, \ldots, n+1\}| = n+1-a,$$

so

$$|X_a| = \binom{n+1-a}{k}.$$

By part (ii) and the ACP, we deduce that

$$\binom{n+1}{k+1} = |X| = \sum_{a=1}^{n-k+1} \binom{n+1-a}{k}.$$

Observing the binomial coefficient terms are exactly

$$\binom{n}{k}, \binom{n-1}{k}, \binom{n-2}{k}, \ldots, \binom{k}{k},$$

we see that this may be rewritten as

$$\binom{n+1}{k+1} = \sum_{\ell=k}^{n} \binom{\ell}{k},$$

as desired.

PROBLEM 2. In this problem we will answer the following question: how many ways are there to write a nonnegative integer m as a sum of r positive integer summands? (We decree that the order of the addends matters, so $3+1$ and $1+3$ are two different representations of 4 as a sum of 2 nonnegative integers.)

(a) Experiment with small cases: let $m = 1, 2, 3, 4$ and $1 \leq r \leq m$.

(b) Develop a conjecture.

(c) Prove your conjecture.

SOLUTION: After playing around for a while, one comes to the conclusion that $\binom{m-1}{r-1}$ gives the desired count. For instance, we can represent 5 as the sum of 3 positive integers as $3+1+1$, $1+3+1$, $1+1+3$, $2+2+1$, $2+1+2$, or $1+2+2$, and $6 = \binom{4}{2}$.

A nice argument for this is given by the Balls and Walls method. Imagine that we have m balls in a row. In order to represent m as a sum of r positive integers, we can place $r-1$ walls in the spaces between the balls, taking care to not place two or more walls in a single gap. For example, the sum $7 = 1 + 3 + 2 + 1$ is represented by

$$\bullet | \bullet \bullet \bullet | \bullet \bullet | \bullet .$$

There is clearly a bijection between such ball-wall configurations and the sums we are counting, and each ball-wall configuration is specified by choosing $r - 1$ spots to places walls amongst the $m - 1$ gaps between balls; this number is, of course, $\binom{m-1}{r-1}$.

PROBLEM 3.

(a) Compute the sums

$$\binom{0}{0}^2$$
$$\binom{1}{0}^2 + \binom{1}{1}^2$$
$$\binom{2}{0}^2 + \binom{2}{1}^2 + \binom{2}{2}^2$$
$$\binom{3}{0}^2 + \binom{3}{1}^2 + \binom{3}{2}^2 + \binom{3}{3}^2$$
$$\binom{4}{0}^2 + \binom{4}{1}^2 + \binom{4}{2}^2 + \binom{4}{3}^2 + \binom{4}{4}^2$$
$$\binom{5}{0}^2 + \binom{5}{1}^2 + \binom{5}{2}^2 + \binom{5}{3}^2 + \binom{5}{4}^2 + \binom{5}{5}^2$$

by hand and develop a conjecture regarding the value of

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n-1}^2 + \binom{n}{n}^2.$$

SOLUTION: we compute

$$\binom{0}{0}^2 = 1$$
$$\binom{1}{0}^2 + \binom{1}{1}^2 = 2$$
$$\binom{2}{0}^2 + \binom{2}{1}^2 + \binom{2}{2}^2 = 6$$
$$\binom{3}{0}^2 + \binom{3}{1}^2 + \binom{3}{2}^2 + \binom{3}{3}^2 = 20$$
$$\binom{4}{0}^2 + \binom{4}{1}^2 + \binom{4}{2}^2 + \binom{4}{3}^2 + \binom{4}{4}^2 = 70$$
$$\binom{5}{0}^2 + \binom{5}{1}^2 + \binom{5}{2}^2 + \binom{5}{3}^2 + \binom{5}{4}^2 + \binom{5}{5}^2 = 252$$

Suspiciously and amazingly, these appear in the center column of Pascal's triangle as the numbers of the form $\binom{2n}{n}$. We conjecture that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n-1}^2 + \binom{n}{n}^2 = \binom{2n}{n}$$

and note that this matches the cases computed above.

(b) Use the binomial theorem to prove your conjecture. [*Hint*: We have the identity $(1+y)^{2n} = (1+y)^n(1+y)^n$. Therefore, if we expand either side and find the coefficient of y^n, we will get the same number. Use the binomial theorem to find the coefficient of y^n in $(1+y)^{2n}$. Next apply the binomial theorem to $(1+y)^n$ and use the result to find the coefficient of y^n in $(1+y)^n(1+y)^n$.]

SOLUTION: Let y be a variable. By the binomial theorem

$$(1+y)^{2n} = \sum_{i=0}^{2n} \binom{2n}{i} y^i.$$

In particular, the coefficient of y^n in this polynomial is $\binom{2n}{n}$.

We also have $(1+y)^{2n} = (1+y)^n(y+1)^n$, and applying the binomial theorem to each factor results in

$$(1+y)^{2n} = \left(\sum_{i=0}^{n} \binom{n}{i} y^i\right)\left(\sum_{j=0}^{n} \binom{n}{j} y^{n-j}\right).$$

When we expand this product, we get a term contributing to y^n when $i+n-j = n$, *i.e.* when $i = j$. Thus the coefficient of y^n is $\sum_{i=0}^{n} \binom{n}{i}^2$, and this must equal our alternate computation of the coefficient, $\binom{2n}{n}$.

(c) Give a combinatorial argument proving your conjecture. [*Hint*: Split a set of size $2n$ into two pieces of size n, and then start building size n subsets of the original set.]

SOLUTION: Suppose $|A| = 2n$ and then color half its elements blue and half its elements red. (We can do that!) To get a size n subset of A, we can choose a blue elements and b red elements where $a+b = n$. For fixed a, there are $\binom{n}{a}\binom{n}{b}$ ways to do this. Since $b = n-a$, we have $\binom{n}{b} = \binom{n}{n-a} = \binom{n}{a}$, and so $\binom{n}{a}\binom{n}{b} = \binom{n}{a}^2$. Letting a vary from 0 to n, we see that in sum we have

$$\begin{aligned}\binom{2n}{n} &= \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{n}\binom{n}{0}\\ &= \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2,\end{aligned}$$

as desired.