PROBLEM 1. Consider the following relations on the set $\mathbb{R}$ of real numbers: inequality ($\neq$), strictly greater than ($>$), and less than or equal to ($\leq$). Determine which (if any) of the three properties of an equivalence relation these relations have:

relation	reflexivity	symmetry	transitivity
$\neq$			
$>$			
$\leq$			

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SOLUTION:

relation	reflexivity	symmetry	transitivity
$\neq$		✓	
$>$			✓
$\leq$	✓		✓

.

The relation $\neq$ is not reflexive ($a \neq a$ is false), is symmetric (if $a \neq b$ then $b \neq a$), and is not transitive ($0 \neq 1$ and $1 \neq 0$, but $0 \neq 0$ is false).

The relation $>$ is not reflexive ($a > a$ is false), is not symmetric ($1 > 0$ but it is not the case that $0 > 1$), and is transitive.

The relation $\leq$ is reflexive, is not symmetric, and is transitive.

PROBLEM 2. Consider the relation $\sim$ on $\mathbb{R}$ such that $x \sim y$ if and only if $x - y$ is an integer.

Recall that for $\simeq$ an equivalence relation on set X, $X/\simeq$ is the set of equivalence classes for $\simeq$.

(a) Give a formal proof (following our template) that $\sim$ is an equivalence relation.

(b) Draw the real number line, choose a point, and draw that point's equivalence class. Repeat for several points.

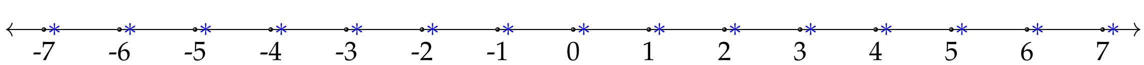
(c) What does a generic element of $\mathbb{R}/\sim$ look like? Does $\mathbb{R}/\sim$ has a natural "shape"?

SOLUTION: We check the properties one by one, beginning with reflexivity: if $x \in \mathbb{R}$, then $x - x = 0 \in \mathbb{Z}$, so $x \sim x$. For symmetry, suppose $x \sim y$, meaning that $x - y \in \mathbb{Z}$. Then $y - x = -(x - y)$ is an integer as well, so $y \sim x$. Finally, we check transitivty: if $x \sim y$ and $y \sim z$, then $x - y, y - z \in \mathbb{Z}$. Thus $(x - y) + (y - z) = x - z \in \mathbb{Z}$, so $x \sim z$.

The $\sim$-equivalence class of $x \in \mathbb{R}$ is

$$[x] = \{x + n \mid n \in \mathbb{Z}\}.$$

Here is the equivalence class for π in blue:



It's a circle! Each equivalence class has exactly one representative of the form $[x]$ for $0 \leq x < 1$, but $[0] = [1]$, so in terms of representatives for equivalence classes, starting at 0 and walking left, when you get to 1 you "come full circle" back to 0. We can think of the natural function

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathbb{R}/\sim \\ x &\longmapsto [x] \end{aligned}$$

as an infinite helix (or coiled spring) projecting down onto a circle.

Take a point-set topology course to make this precise!

PROBLEM 3. We place two red and two black checkers on the corners of a square. Say that two configurations are equivalent if one can be rotated to the other.

(a) Check that this is an equivalence relation.

(b) Draw the elements in each equivalence class.

(c) If $\sim$ is a relation on a finite set S, and each equivalence class has the same number k of elements, then the overcounting principle says the number of equivalence classes is $|S|/k$. Why don't these ideas apply to our problem?

SOLUTION: Again, it's fairly "clear" that this is an equivalence relation. (But check!) In order to enumerate the equivalence classes, we will consider a word using RRBB to have first letter corresponding to the color in the northwest corner, second letter corresponding to northeast corner, third corresponding to southeast, and fourth corresponding to southwest. Each word has up to four potentially distinct rotations:

$$\begin{aligned} RRBB \rightarrow RBBR \rightarrow BBRR \rightarrow BRRB \\ RBRB \rightarrow BRBR \rightarrow RBRB \rightarrow BRBR \end{aligned}$$

We stop here because we've enumerated all the words in RRBB, but note that words are repeated in the second set of rotations. The equivalence classes are in fact

$$\{RRBB, RBRB, BBRR, BRRB\} \text{ and } \{RBRB, BRBR\}.$$

Thus, there are two equivalence classes, i.e., two ways to color the corners up to rotational symmetry. Let S be the set of all words of length four containing two Rs and two Bs. Then $|S| = 4!/(2!2!) = 6$. If both equivalence classes had three elements, then we could have gotten our answer by the overcounting princile as $|S|/2$ where 2 denotes the number of equivalence classes. Since our equivalence classes do not have the same cardinality, that argument does not apply.

PROBLEM 4. A total of n Terraneans and n Gethenians* attend a meeting and sit around a round table. If Terraneans and Gethenians alternate seats, in how many ways may they be seated up to rotation? Discuss your solution in terms of an equivalence relation and equal-sized equivalence classes.

* From the planet Gethen, which is the setting of *The Left Hand of Darkness* by Ursula K. Le Guin.

SOLUTION: We present two solutions. For the first, label the seats $1, \ldots, 2n$. Put Gethenians in seats $1, 3, \ldots, 2n-1$ and put Terraneans in seats $2, 4, \ldots, 2n$. There are $n! \cdot n! = (n!)^2$ ways to do so. But we could have also put Gethenians in the even seats and Terraneans in the odd seats, so there are in fact $2(n!)^2$ total valid seatings (by the ACP). Declare two such seatings equivalent if one can be rotated to obtain the other. (We take it as given that this forms an equivalence relation, but it's good practice to check the conditions.) There are $2n$ such rotations, so there are

$$\frac{2(n!)^2}{2n} = (n-1)!n!$$

seating arrangements.

Now for the second argument: Without loss of generality, assume that one of the Gethenians is named Estraven. We can choose a unique representative of each rotational equivalence class of seatings by selecting the seating with Estraven in seat 1. The remaining Gethenians must then go in seats $3, 5, \ldots, 2n-1$, and the Terraneans can sit freely in seats $2, 4, \ldots, 2n$. This gives a direct count of $(n-1)!n!$.