PROBLEM 1. Suppose $k, n \in \mathbb{N}$ with $k \leq n$. Give two proofs that

$$\binom{n}{k} = \binom{n}{n-k}.$$

The first proof should be algebraic, using the defining formulas. The second should explain why both sides of the equality count the same thing.

PROBLEM 2. (Revisiting a problem from week 1.) Let $a, b \in \mathbb{N}$. Prove that the number of NE lattice paths from $(0, 0)$ to (a, b) is

$$\binom{a+b}{a}.$$

(Note that by problem 1, this is equal to $\binom{a+b}{b}$.)

PROBLEM 3. Show that there are $1,098,240$ one-pair poker hands.[*]

[*] Poker is played with the standard deck of 52 cards, with four suits, and each suit containing 13 denominations. A poker hand consists of 5 cards. A one-pair poker hand is a hand that contains two of the same denomination and no other repeated denomination. For example, a hand with two kinds, one ace, one queen and one jack is a one-pair poker hand.

PROBLEM 4. There are 24 students in this class. How many ways can I pair you up (split you into groups of 2)? How many ways can I split you into groups of three?

Challenge

Challenge problems are optional and should only be attempted after completing the previous problems.

Ten ants are dropped in random positions on a meter-long stick. Some of these ants are initially traveling to the left and some are traveling to the right, but all travel at one meter/minute. When two ants meet, they bounce off of each other and change their directions (instantaneously). When an ant reaches the end of the stick, it walks off, never to return. What is the maximal amount of time (over all possible initial conditions) before the stick to be ant free? Characterize all initial conditions that achieve this maximal time. (If you have seen this problem before, do not spoil it for others in your group!)