PROBLEM 1. Suppose $k, n \in \mathbb{N}$ with $k \leq n$. Give two proofs that

$$\binom{n}{k} = \binom{n}{n-k}.$$

The first proof should be algebraic, using the defining formulas. The second should explain why both sides of the equality count the same thing.

SOLUTION: *Algebraic proof.* We have

$$\binom{n}{k} := \frac{n!}{k!(n-k)!} = \frac{n!}{(n-(n-k))!(n-k)!} = \frac{n!}{(n-k)!(n-(n-k))!} =: \binom{n}{n-k}.$$

Combinatorial proof. Think of the binomial $\binom{n}{k}$ as the number of ways of choosing a k-person committee from a set of n people. Choosing a set of k people to be on the committee is the same as choosing $n-k$ people to not be on the committee, and there are $\binom{n}{n-k}$ ways to do that.

More formally, let X_k be the set of all k-element subsets of $[n] := \{1, \ldots, n\}$, and let X_{n-k} be the set of all $(n-k)$-element subsets of $[n]$. Then we know that

$$|X_k| = \binom{n}{k} \quad \text{and} \quad |X_{n-k}| = \binom{n}{n-k}.$$

These numbers are the same since there is a bijection:

$$\begin{aligned} X_k &\to X_{n-k} \\ S &\mapsto [n] \setminus S. \end{aligned}$$

PROBLEM 2. Let $a, b \in \mathbb{N}$. Prove that the number of NE lattice paths from $(0,0)$ to (a,b) is

$$\binom{a+b}{a}.$$

(Note that by problem 1, this is equal to $\binom{a+b}{b}$.)

SOLUTION: As we have seen, a lattice path from $(0,0)$ to (a,b) is the same as a word in the letters N and E of length $a+b$ using the letter N a total of a times and the letter E a total of b times. To determine such a word, we just need to decide which of the $a+b$ letters are Ns. The rest of the letters will then be Es. There are $\binom{a+b}{b}$ to make those choices. Similarly, we could choose which of the $a+b$ letters are Es, letting the rest of the letters be Ns. There are $\binom{a+b}{b}$ ways to do that.

PROBLEM 3. Show that there are 1,098,240 one-pair poker hands.*

SOLUTION: First choose four of the 13 denominations ($\binom{13}{4}$ choices). Then choose one of these four denominations from which to choose the pair ($4 = \binom{4}{1}$ choices). Then choose two cards from that denomination ($6 = \binom{4}{2}$ choices). From each of the other three denominations, we must choose one card (4 choices for each). By the multiplicative counting principle, the number of one-pair poker hands is

$$\binom{13}{4} \cdot 4 \cdot 6 \cdot 4^3 = 1098240.$$

* Poker is played with the standard deck of 52 cards, with four suits, and each suit containing 13 denominations. A poker hand consists of 5 cards. A one-pair poker hand is a hand that contains two of the same denomination and no other repeated denomination. For example, a hand with two kinds, one ace, one queen and one jack is a one-pair poker hand.

PROBLEM 4. There are 24 students in this class. How many ways can I pair you up (split you into groups of 2)? How many ways can I split you into groups of three?

SOLUTION: There are $\binom{24}{2}$ ways of choosing the first pair, once this pair is chosen, there are $\binom{22}{2}$ ways to chose the next pair, and so on. Note that this chooses the pairs in a specific order, but since I don't care about the order of the pairs, we have overcounted by 12!, which is the total number of permutations for the pairs. Thus we have

$$\frac{\binom{24}{2} \cdot \binom{22}{2} \cdots \binom{2}{2}}{12!}.$$

Alternatively, we can achieve the same answer as follows: order the students alphabetically. The first student on the list can be paired up with 23 other students. Now take the first student and its pair out of the list. The next student on the list can then be paired up with 21 other students. Continuing this way, we note that there are

$$23 \cdot 21 \cdot 19 \ldots 3 \cdot 1$$

ways of pairing the students up (it is good to check the two answers agree).

For the triples, we proceed just like in the first solution for pairs. There are $\binom{24}{3}$ ways of choosing the first triple, $\binom{21}{3}$ for the second triple, and so on. This chooses the triples in order, so we are avourcounting by 8!. Thus, the number of ways to split up the class into groups of three is

$$\frac{\binom{24}{3} \cdot \binom{21}{3} \cdots \binom{3}{3}}{8!}.$$

Challenge

Challenge problems are optional and should only be attempted after completing the previous problems.

Ten ants are dropped in random positions on a meter-long stick. Some of these ants are initially traveling to the left and some are traveling to the right, but all travel at one meter/minute. When two ants meet, they bounce off of each other and change their directions (instantaneously). When an ant reaches the end of the stick, it walks off, never to return. What is the maximal amount of time (over all possible initial conditions) before the stick to be ant free? Characterize all initial conditions that achieve this maximal time. (If you have seen this problem before, do not spoil it for others in your group!)

SOLUTION: Imagine two ants heading in opposite directions and just about to collide. To keep things straight, say one ant is blue and the other ant is red. According to the problem, when the ants collide, each changes direction. Another way of thinking of this, though, is that the ants do not change directions but that they change colors. Or, dropping the colors, we can think of the ants as just changing identities. Thinking in that way, the ants never change directions—they just keep walking until they reach the end of the meter stick. Since they are walking at one meter/minute, the maximal time an ant can spend on the stick is one minute, and this happens exactly when the ant needs to walk the full length of the stick.