Recall that for $n \in \mathbb{Z}_{\geq 1}$, we denote $[n] = \{1, \ldots, n\}$. Note that the symbol $\to$ is used between sets (the domain and codomain), whereas the symbol $\mapsto$ means "maps to", and is used between elements.

PROBLEM 1. For each of the following, decide:

- Does the mapping give a well-defined function? (If not, why?)

If so:

- Graph the function.
- Is the function injective, surjective, both, or neither?
- Is the function invertible? If so, what is the inverse?

$$\begin{array}{rcl} f\colon [5] & \to & [3] \\ 1 & \mapsto & 1 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 1 \\ 4 & \mapsto & 2 \\ 5 & \mapsto & 1 \end{array} \qquad \begin{array}{rcl} g\colon [5] & \to & [3] \\ 1 & \mapsto & 1 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 3 \\ 4 & \mapsto & 2 \\ 5 & \mapsto & 1 \end{array} \qquad \begin{array}{rcl} h\colon [5] & \to & [3] \\ 1 & \mapsto & 1 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 3 \\ 4 & \mapsto & 5 \\ 5 & \mapsto & 4 \end{array}$$

$$\begin{array}{rcl} \varphi\colon [3] & \to & [4] \\ 1 & \mapsto & 3 \\ 2 & \mapsto & 1 \\ 3 & \mapsto & 2 \end{array} \qquad \begin{array}{rcl} \psi\colon [3] & \to & [4] \\ 1 & \mapsto & 3 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 3 \end{array} \qquad \begin{array}{rcl} \sigma\colon [3] & \to & [3] \\ 1 & \mapsto & 3 \\ 2 & \mapsto & 1 \\ 3 & \mapsto & 2 \end{array} \qquad \begin{array}{rcl} \tau\colon [3] & \to & [3] \\ 1 & \mapsto & 3 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 3 \end{array}$$

PROBLEM 2. Which ordered pairs of functions from Problem 1 are composable (for which functions $a$ and $b$ is $a \circ b$ defined)? Compute the composites for two or three of these examples. [Hint: For example, $\varphi \circ f$ is defined, but $f \circ \varphi$ is not. Caution: $\varphi \circ \varphi$ is not defined—why?]

PROBLEM 3. Let $A$ and $B$ be finite sets, and let $f\colon A \to B$ be a function.

(a) Suppose $f$ is injective. What can you say about the cardinalities of $A$, $\operatorname{im}(f)$, and $B$? Why?

(b) Suppose $f$ is surjective. What can you say about the cardinalities of $A$, $\operatorname{im}(f)$, and $B$? Why?

PROBLEM 4. Let $n, k$ be integers such that $1 \leq k \leq n$, and consider the following two sets.

$$A = \{X \subseteq [n] \mid |X| = k \text{ and } n \in X\},$$
$$B = \{Y \subseteq [n-1] \mid |Y| = k - 1\}.$$

Prove that $|A| = |B|$ by producing a bijection $f\colon A \to B$. You need to define the function $f$ and prove that it is a bijection, either by proving it has a two-sided inverse, or proving that it is injective and surjective.

PROBLEM 5. Define a function $f\colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ by the piecewise formula

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{-1-n}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Show that $f$ is a bijection by finding a function $g\colon \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ that is a two-sided inverse to $f$. [Hint: Start by computing $f(n)$ for $n = 0, 1, 2, 3 \ldots$. Then write out $g(k)$ for $k = 0, \pm 1, \pm 2 \ldots$, using $f(n) = k$ means $g(k) = n$. Then try to write a piecewise formula.]

| $n$: | 0 | 1 | 2 | 3 | $\cdots$ |
|---|---|---|---|---|---|
| $f(n)$: | | | | | |

| $k$: | $\cdots$ | $-2$ | $-1$ | 0 | 1 | 2 | $\cdots$ |
|---|---|---|---|---|---|---|---|
| $g(k)$: | | | | | | | |

PROBLEM 6. Consider the function $g\colon \mathbb{Z} \to \mathbb{Z}$ given by

$$g(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Determine whether or not $g$ is injective, and whether or not $g$ is surjective. Prove your answers.

## *Challenge*

Challenge problems are optional and should only be attempted after completing the previous problems.

PROBLEM. Let $A$ and $B$ be sets, and let $f\colon A \to B$ be a function. For $X \subseteq A$, the *image of $X$ along $f$* is

$$f(X) = \{f(x) \mid x \in X\};$$

and for $Y \subseteq B$, the *preimage of $Y$ along $f$* is

$$f^{-1}(Y) = \{x \in A \mid f(x) \in Y\}.$$

The notation $f^{-1}(Y)$ isn't meant to imply that $f$ is an invertible function: it's just the set defined above, and might even be empty!

Now, define two new functions

$$\begin{aligned} F\colon 2^A &\to 2^B \\ X &\mapsto f(X) \end{aligned} \qquad \text{and} \qquad \begin{aligned} G\colon 2^B &\to 2^A \\ Y &\mapsto f^{-1}(Y) \end{aligned}.$$

(a) Do some examples. What are $F$ and $G$ for $\tau$ in Problem 1? How does your answer change if working with $\psi$ or $\sigma$ instead?

(b) Draw cartoons illustrating $f(X)$ and $f^{-1}(Y)$.

(c) Is there any relationship between whether or not $f$ is surjective and whether or not $F$ is surjective? What about injectivity? What about $G$?

(d) Let $X_1, X_2 \subseteq A$ and $Y_1, Y_2 \subseteq B$. Explore each of the following statements: first convince yourself of their truth, and then prove the result.

$$\begin{aligned} F(X_1 \cup X_2) &= F(X_1) \cup F(X_2) \\ F(X_1 \cap X_2) &\subseteq F(X_1) \cap F(X_2) \\ G(Y_1 \cup Y_2) &= G(Y_1) \cup G(Y_2) \\ G(Y_1 \cap Y_2) &= G(Y_1) \cap G(Y_2) \end{aligned}$$

For the second statements, give an example showing why we don't have equality. [Hint: Try an example where $f$ is not injective.]

(e) Show that $F(X) \subseteq Y$ if and only if $X \subseteq G(Y)$.