PROBLEM 1. For each of the following, decide:

- Does the mapping give a well-defined function? (If not, why?)

If so:

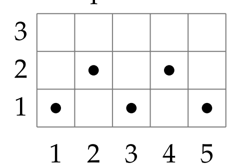
- Graph the function.
- Is the function injective, surjective, both, or neither?
- Is the function invertible? If so, what is the inverse?

Recall that for $n \in \mathbb{Z}_{\geq 1}$, we denote $[n] = \{1, \ldots, n\}$. Note that the symbol $\to$ is used between sets (the domain and codomain), whereas the symbol $\mapsto$ means "maps to", and is used between elements.

SOLUTION:

$$\begin{array}{rcl} f\colon [5] & \to & [3] \\ 1 & \mapsto & 1 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 1 \\ 4 & \mapsto & 2 \\ 5 & \mapsto & 1 \end{array}$$

Graph:

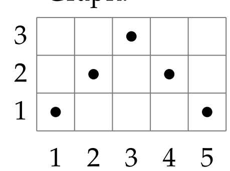


The map f is well-defined, but is neither injective (for example, $f(1) = f(3)$) nor surjective (for example, $3 \in [3] \setminus \operatorname{im}(f)$).

. . . .

$$\begin{array}{rcl} g\colon [5] & \to & [3] \\ 1 & \mapsto & 1 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 3 \\ 4 & \mapsto & 2 \\ 5 & \mapsto & 1 \end{array}$$

Graph:

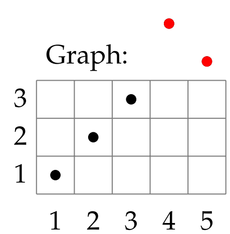


The map g is well-defined and surjective, but is not injective (for example, $g(1) = g(5)$).

. . . .

$$\begin{array}{rcl} h\colon [5] & \to & [3] \\ 1 & \mapsto & 1 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 3 \\ 4 & \mapsto & 5 \\ 5 & \mapsto & 4 \end{array}$$

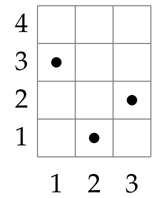
Graph:



The map h is not well-defined, since $h(4) = 5$ but $5 \notin [3]$.

$$\begin{array}{rcl} \varphi\colon [3] & \to & [4] \\ 1 & \mapsto & 3 \\ 2 & \mapsto & 1 \\ 3 & \mapsto & 2 \end{array}$$

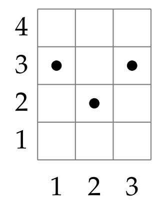
Graph:



The map φ is well-defined and injective, but is not surjective (for example, $4 \in [4] \smallsetminus \operatorname{im}(\varphi)$).

$$\begin{array}{rcl} \psi\colon [3] & \to & [4] \\ 1 & \mapsto & 3 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 3 \end{array}$$

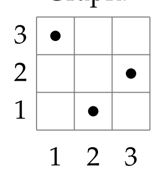
Graph:



The map ψ is well-defined, but is neither injective (for example, $\psi(1) = \psi(3)$) nor surjective (for example, $4 \in [4] \smallsetminus \operatorname{im}(\psi)$).

$$\begin{array}{rcl} \sigma\colon [3] & \to & [3] \\ 1 & \mapsto & 3 \\ 2 & \mapsto & 1 \\ 3 & \mapsto & 2 \end{array}$$

Graph:

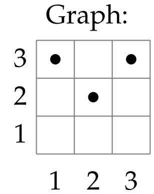


The map σ is well-defined, injective, and surjective. Thus σ is bijective, and therefore invertible with

$$\begin{array}{rcl} \sigma^{-1}\colon [3] & \to & [3] \\ 1 & \mapsto & 2 \\ 2 & \mapsto & 3 \\ 3 & \mapsto & 1 \end{array}$$

$$\begin{array}{rcl} \tau\colon [3] & \to & [3] \\ 1 & \mapsto & 3 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 3 \end{array}$$

Graph:

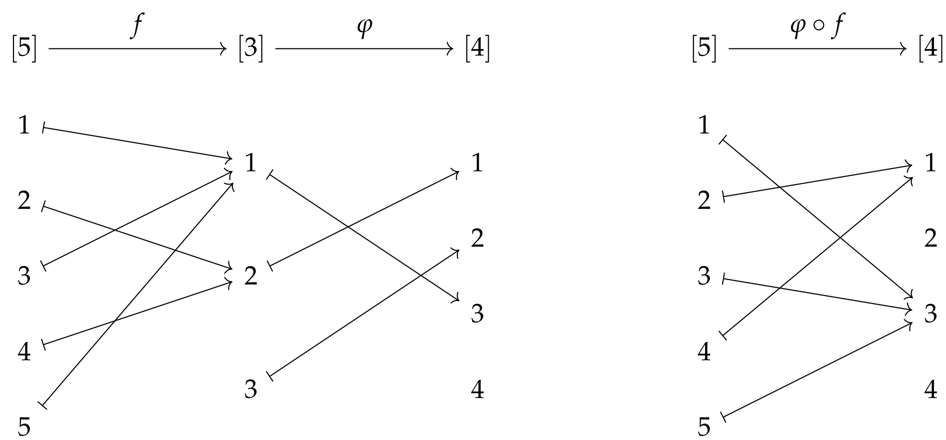


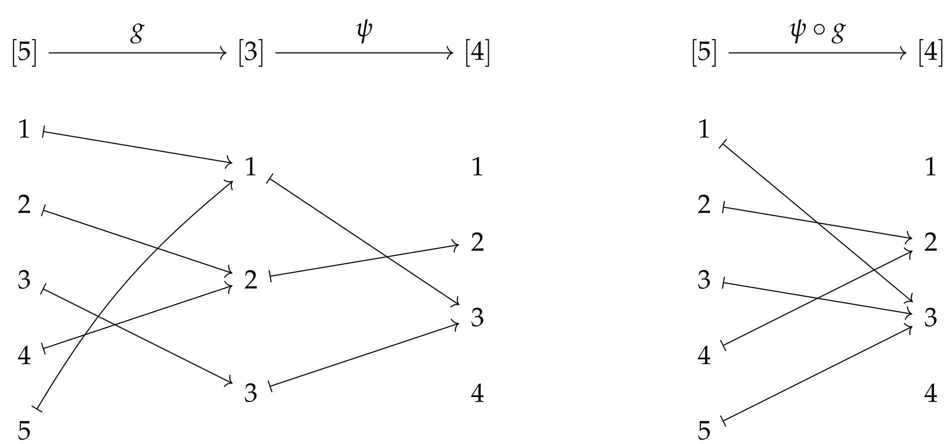
The map τ is well-defined, but is neither injective (for example, $\tau(1) = \tau(3)$) nor surjective (for example, $1 \in [3] \smallsetminus \operatorname{im}(\tau)$).

PROBLEM 2. Which ordered pairs of functions from Problem 1 are composable (for which functions a and b is $a \circ b$ defined)? Compute the composites for two or three of these examples. [Hint: For example, $\varphi \circ f$ is defined, but $f \circ \varphi$ is not. Caution: $\varphi \circ \varphi$ is not defined–why?]

SOLUTION: For two functions $a\colon A \to B$ and $b\colon C \to D$, we have that $a \circ b$ is defined if and only if $D = A$. (Even if $\mathrm{im}(b) \subset D$, we don't define $a \circ b$ unless the *codomain* of b matches the domain of a. So $a \circ b$ is defined for any $a \in \{\varphi, \psi, \sigma, \tau\}$ and $b \in \{f, g, \sigma, \tau\}$.

A couple of examples out of these 16 possibilities:





PROBLEM 3. Let A and B be finite sets, and let $f\colon A \to B$ be a function.

(a) Suppose f is injective. What can you say about the cardinalities of A, $\operatorname{im}(f)$, and B? Why?

(b) Suppose f is surjective. What can you say about the cardinalities of A, $\operatorname{im}(f)$, and B? Why?

SOLUTION:

(a) If f is injective, every element in A has a distinct element in $\operatorname{im}(f)$ corresponding to it, so $|\operatorname{im}(f)| = |A|$. But there may be additional elements in B besides those in $\operatorname{im}(f)$. Thus $|A| = |\operatorname{im}(f)| \leq |B|$.

(b) If f is surjective, $\operatorname{im}(f) = B$, so that every element of B has at least one element from A that maps to it, so $|A| \geq |\operatorname{im}(f)| = |B|$.

PROBLEM 4. Let n, k be integers such that $1 \leq k \leq n$, and consider the following two sets.

$$A = \{X \subseteq [n] \mid |X| = k \text{ and } n \in X\},$$
$$B = \{Y \subseteq [n-1] \mid |Y| = k-1\}.$$

Prove that $|A| = |B|$ by producing a bijection $f\colon A \to B$. You need to define the function f and prove that it is a bijection, either by proving it has a two-sided inverse, or proving that it is injective and surjective.

SOLUTION: We define the function $f\colon A \to B$ by $f(X) = X \setminus \{n\}$. Since X is a subset of $[n]$ that contains n, then $f(X)$ is a subset of $[n-1]$, and moreover, given that X has k elements, then we know that $f(X)$ has $k-1$ elements. This shows that $f(X) \in B$.

To prove that f is a bijection we construct a two-sided inverse, $g\colon B \to A$. For $Y \in B$, we define $g(Y) = Y \cup \{n\}$. Note that since $Y \subseteq [n-1]$, we then know that $g(Y) \subseteq [n]$, and given that Y has $k-1$ elements, by construction $g(Y)$ has k elements. This shows that $g(Y) \in A$. Now note that f takes a set that contains n and removes n from it, and g takes a set that does not contain n and includes n in it, thus showing that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. Therefore f is a bijection.

PROBLEM 5. Define a function $f\colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ by the piecewise formula

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{-1-n}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Show that f is a bijection by finding a function $g\colon \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ that is a two-sided inverse to f. [Hint: Start by computing $f(n)$ for $n = 0, 1, 2, 3 \ldots$. Then write out $g(k)$ for $k = 0, \pm 1, \pm 2 \ldots$, using $f(n) = k$ means $g(k) = n$. Then try to write a piecewise formula.]

n:	0	1	2	3	$\cdots$
$f(n)$:					

k:	$\cdots$	-2	-1	0	1	2	$\cdots$
$g(k)$:							

SOLUTION: First, let's compute a few values of f to get a sense of what's going on. Breaking into even and odd cases, we get some of the following data points.

even n:

n:	0	2	4	6	$\cdots$
$f(n)$:	0	1	2	3	$\cdots$

odd n:

n:	1	3	5	7	$\cdots$
$f(n)$:	-1	-2	-3	-4	$\cdots$

In particular, it looks like all every element of $\mathbb{Z}_{\geq 0}$ gets mapped to by some even n; and every element of $\mathbb{Z}_{<0}$ gets mapped to by some odd n. Turning this around, the inverse must satisfy the following:

k:	$\cdots$	-4	-3	-2	-1	0	1	2	3	$\cdots$
$g(k)$:	$\ldots$	7	5	3	1	0	2	4	6	$\cdots$

So we should break the definition of g into the cases where $k \leq -1$ and $k \geq 0$. For $k \leq -1$, we're in the image of the odd branch of f; so solve

$$\frac{-1-n}{2} = k \quad \text{for } n \text{ to get} \quad g(k) = n = -(2k+1).$$

Checking against our data verifies this formula. And for $k \geq 0$, we're in the even branch of f; so solve

$$\frac{n}{2} = k \quad \text{for } n \text{ to get} \quad g(k) = n = 2k.$$

Again, checking against our data verifies this formula.

Putting it all together, define $g\colon \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ by the piecewise formula

$$g(k) = \begin{cases} 2k & \text{if } k \geq 0, \\ -(2k+1) & \text{if } k \leq 1. \end{cases}$$

A piecewise algebraic check shows that $f \circ g = \mathrm{id}_{\mathbb{Z}}$ and $g \circ f = \mathrm{id}_{\mathbb{Z}_{\geq 0}}$.

PROBLEM 6. Consider the function $g\colon \mathbb{Z} \to \mathbb{Z}$ given by

$$g(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Determine whether or not g is injective, and whether or not g is surjective. Prove your answers.

SOLUTION: Again, it's good to start by collecting some data:

even n:

n :	$\cdots$	-4	-2	0	2	4	$\cdots$
$g(n)$:	$\cdots$	-2	-1	0	1	2	$\cdots$

odd n:

n :	$\cdots$	-3	-1	1	3	$\cdots$
$g(n)$:	$\cdots$	-1	0	1	2	$\cdots$

From this, we can see that g is *not* injective; for example,

$$g(1) = 1 = g(2).$$

On the other hand, g is surjective: for any $k \in \mathbb{Z}$, we would like to show that $k \in \operatorname{im}(g)$. Taking a hind from our "even n" data above, note that $2k \in \mathbb{Z}$ is even. Hence,

$$g(2k) = \frac{2k}{2} = k \in \operatorname{im}(g).$$

Challenge

Challenge problems are optional and should only be attempted after completing the previous problems.

PROBLEM. Let A and B be sets, and let $f\colon A \to B$ be a function. For $X \subseteq A$, the *image of X along f* is

$$f(X) = \{f(x) \mid x \in X\};$$

and for $Y \subseteq B$, the *preimage of Y along f* is

$$f^{-1}(Y) = \{x \in A \mid f(x) \in Y\}.$$

(The notation $f^{-1}(Y)$ isn't meant to imply that f is an invertible function: it's just the set defined above, and might even be empty!)

Now, define two new functions

$$\begin{aligned} F\colon 2^A &\to 2^B \\ X &\mapsto f(X) \end{aligned} \quad \text{and} \quad \begin{aligned} G\colon 2^B &\to 2^A \\ Y &\mapsto f^{-1}(Y) \end{aligned}.$$

(a) Do some examples. What are F and G for τ in Problem 1? How does your answer change if working with ψ or σ instead?

SOLUTION: For τ, we have

F:	$2^{[3]}$	$\to$	$2^{[3]}$		G:	$2^{[3]}$	$\to$	$2^{[3]}$
	$\varnothing$	$\mapsto$	$\varnothing$			$\varnothing$	$\mapsto$	$\varnothing$
	$\{1\}$	$\mapsto$	$\{3\}$			$\{1\}$	$\mapsto$	$\varnothing$
	$\{2\}$	$\mapsto$	$\{2\}$			$\{2\}$	$\mapsto$	$\{2\}$
	$\{3\}$	$\mapsto$	$\{3\}$	and		$\{3\}$	$\mapsto$	$\{1,3\}$
	$\{1,2\}$	$\mapsto$	$\{2,3\}$			$\{1,2\}$	$\mapsto$	$\{1,2,3\}$
	$\{1,3\}$	$\mapsto$	$\{3\}$			$\{1,3\}$	$\mapsto$	$\{3\}$
	$\{2,3\}$	$\mapsto$	$\{2,3\}$			$\{2,3\}$	$\mapsto$	$\{1,2,3\}$
	$\{1,2,3\}$	$\mapsto$	$\{2,3\}$			$\{1,2,3\}$	$\mapsto$	$\{1,2,3\}$

(b) Draw cartoons illustrating $f(X)$ and $f^{-1}(Y)$.

(c) Is there any relationship between whether or not f is surjective and whether or not F is surjective? What about injectivity? What about G?

SOLUTION: If F is surjective, then there is at least one $X \subseteq A$ where $f(X) = B$, so f is surjective. And if f is surjective, then so is F: let $Y \subseteq B$, and for each $y \in Y$, there is at least one a_y for which $f(a_y) = y$. So $X = \{a_y \mid y \in Y\}$ satisfies $f(X) = Y$.

Next, suppose F is injective. Then for any $a_1 \neq a_2$ in A, we have $\{a_1\} \neq \{a_2\}$; and hence $F(\{a_1\}) \neq F(\{a_2\})$, so that $f(a_1) \neq f(a_2)$. So f is also injective. On the other hand, if f is injective, then for any $a_1 \neq a_2$, we have $f(a_1) \neq f(a_2)$. And for $X_1 \neq X_2$ in 2^A, there is at least one element that X_1 and X_2 do not have in common; and hence there is at least one element that $f(X_1)$ and $f(X_2)$ do not have in common. So $f(X_1) \neq f(X_2)$. Thus F is injective.

Moving on to G: if G is surjective, then in particular, $\{a\} \in \text{im}(G)$ for every $a \in A$. But since f is well-defined, if

$$\{a\} = G(Y) = \{x \in A \mid f(x) \in Y\},$$

we must have $|Y| = 1$. So $Y = \{b\}$ for some $b \in B$, and hence a is the only element that maps to b. This is true across all elements of A, so f must be injective. However, f need not be surjective: consider the map $f : [1] \to [2]$ defined by $1 \mapsto 1$. Then $G(\emptyset) = \emptyset$ and $G(\{1\}) = [1]$; and hence G is surjective.

Continuing on, one can show that if f is injective, then G must be surjective; and that G is injective if and only if f is surjective.

(d) Let $X_1, X_2 \subseteq A$ and $Y_1, Y_2 \subseteq B$. Explore each of the following statements: first convince yourself of their truth, and then prove the result.

$$F(X_1 \cup X_2) = F(X_1) \cup F(X_2)$$
$$F(X_1 \cap X_2) \subseteq F(X_1) \cap F(X_2)$$
$$G(Y_1 \cup Y_2) = G(Y_1) \cup G(Y_2)$$
$$G(Y_1 \cap Y_2) = G(Y_1) \cap G(Y_2)$$

For the second statements, give an example showing why we don't have equality.

SOLUTION: We have

$$\begin{aligned} F(X_1 \cup X_2) &= \{f(x) \mid x \in X_1 \cup X_2\} \\ &= \{f(x) \mid x \in X_1 \text{ or } x \in X_2\} \\ &= \{f(x) \mid x \in X_1\} \cup \{f(x) \mid x \in X_2\} \\ &= F(X_1) \cup F(X_2). \end{aligned}$$

Next,

$$\begin{aligned} F(X_1 \cap X_2) &= \{f(x) \mid x \in X_1 \cap X_2\} \\ &= \{f(x) \mid x \in X_1 \text{ and } x \in X_2\}. \end{aligned}$$

In particular, if $y \in F(X_1 \cap X_2)$, then $y = f(x)$ for some $x \in X_1$, so that $y \in F(X_1)$; and $y = f(x)$ for some $x \in X_2$, so that $y \in F(X_2)$. Hence $y \in F(X_1) \cap F(X_2)$, showing that $F(X_1 \cap X_2) \subseteq F(X_1) \cap F(X_2)$.

However, if we consider the example where $f\colon [2] \to [1]$ is defined by $1, 2 \mapsto 1$. Then with $X_1 = \{1\}$ and $X_2 = \{2\}$, we have

$$F(X_1 \cap X_2) = F(\emptyset) = \emptyset$$

but

$$F(X_1) \cap F(X_2) = \{1\} \cap \{1\} = \{1\}.$$

For the third pair, we have

$$\begin{aligned} G(Y_1) \cup G(Y_2) &= \{x_1 \in A \mid f(x_1) \in Y_1\} \cup \{x_2 \in A \mid f(x_2) \in Y_2\} \\ &= \{x \in A \mid f(x) \in Y_1 \text{ or } f(x) \in Y_2\} \\ &= \{x \in A \mid f(x) \in Y_1 \cup Y_2\} \\ &= G(Y_1 \cup Y_2). \end{aligned}$$

And finally,

$$\begin{aligned} G(Y_1) \cap G(Y_2) &= \{x_1 \in A \mid f(x_1) \in Y_1\} \cap \{x_2 \in A \mid f(x_2) \in Y_2\} \\ &= \{x \in A \mid f(x) \in Y_1 \text{ and } f(x) \in Y_2\} \\ &= \{x \in A \mid f(x) \in Y_1 \cap Y_2\} \\ &= G(Y_1 \cap Y_2). \end{aligned}$$

(e) Show that $F(X) \subseteq Y$ if and only if $X \subseteq G(Y)$.