PROBLEM 1. How many words can you make by rearranging the letters of the word *susurrus* if you do not care whether the words make sense?

SOLUTION: First imagine the repeated letters are marked so that they are distinguishable:

$$s_1 u_1 s_2 u_2 r_1 r_2 u_3 s_3.$$

In that case, there are eight choices for the first letter, and for each of these, there are seven choices for the second, and so on. By the multiplicative counting principle, there are 8! words. For each of these words there are 3! rearrangements of the s_i, and for each of these rearrangements, there are 3! rearrangements of the u_i, and finally for each of these, there are 2! rearrangements of the r_i. Dropping the subscripts, we see that each word has been overcounted by a factor of $3! \cdot 3! \cdot 2!$. Therefore, by the overcounting principle, the solution to the original problem is

$$\frac{8!}{3!3!2!} = 560.$$

PROBLEM 2. To form a password, you can either form as sequence of six digits from $\{0, 1, \ldots, 9\}$ or a sequence of four letters from $\{a, \ldots, z\}$.

(a) How many possible passwords are there if no number or letter can be repeated?

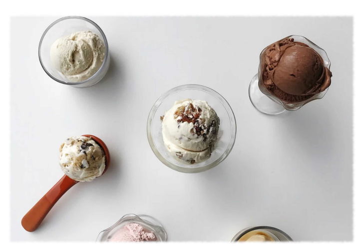
SOLUTION: There are $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = \frac{10!}{4!}$ sequences of distinct digits, and there are $26 \cdot 25 \cdot 24 \cdot 23 = \frac{26!}{22!}$ sequences of distinct letters. Thus, the total number of passwords is

$$\frac{10!}{4!} + \frac{26!}{22!} = 510000.$$

(b) How many if repetitions are allowed?

SOLUTION: If repetitions are allowed, there are 10^6 choices for a numerical password and 26^4 choices for a password made with letters. So in this case, the total number of passwords is

$$10^6 + 26^4 = 1456976.$$



PROBLEM 3. You are constructing a nine-layer ice cream cake and go to Cloud City Ice Cream to pick out the flavors. You decide on the following:

three layers of Dark Chocolate Salted Caramel
one layer of Caramelized Banana
two layers of Earl Grey Blueberry
one layer of Honey Lavender
two layers of Oregon Strawberry.

How many choices do you have for the arrangement of the layers?

SOLUTION: Use the letters d, c, e, h, o to refer to the flavors in the obvious way. Then this problem asks how many words can we make by rearranging the letters *dddceehoo*. Applying the same reasoning we used in Problem 1, the number of choices is

$$\frac{9!}{3!2!2!} = 15120.$$

PROBLEM 4. Five couples go to the theater and sit in the first row, which conveniently has exactly ten seats. How many ways can these people be seated if couples must sit together?

SOLUTION: By the MCP, there are 5! ways to arrange the couples, and then for each couple, there are two choices for their arrangement. Therefore, the total numbers of possibilities is

$$2^5 \cdot 5! = 3840.$$

Another way to reason about this problem is as follows: There are ten choices for who sits in the first seat. This person's partner must sit in the second seat. That leaves eight choices for the third seat, and that person's partner sits in the fourth seat. Continuing like this, we see that the solution is

$$10 \cdot 8 \cdot 6 \cdot 4 \cdot 2 = 3840 \quad (= 2^5 \cdot 5!).$$

Challenge

Challenge problems are optional and should only be attempted after completing the previous problems.

PROBLEM. How many ways are there to choose an ordered pair of subsets (A, B) from $\{0, 1, \cdots, 9\}$ such that $|A \cap B| = 1$?

SOLUTION: There are ten choices for the element in $A \cap B$. For each of the remaining nine elements, there are three choices: the element is in A, it is in B, or it is in neither. By the multiplicative counting principle, the solution is

$$10 \cdot 3^9 = 196830.$$